Graduate Course

B.A. (PROGRAMME) 1 YEAR MATHEMATICS
ALGEBRA & CALCULUS

PART-B : CALCULUS

SM-4

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Editor:
Dr. S.K. Verma

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INTRODUCTION
One of the earliest applications of Differential calculus was the study of curves by means of their tangents and normals. In this lesson we shall give a geometrical interpretation of the differential coefficient \( \frac{dy}{dx} \) at a point on a curve. Subsequently we will find out the equations of the tangent and normal to a curve whose equation is given.

1. (a) Definition of the Tangent to a Curve at a Given Point
Let \( P \) be a given point on a curve, and \( Q \) another point on the curve close to \( P \). Then as \( Q \) tends to \( P \), the secant \( PQ\) in general tends to a straight line \( PT \), say, which is called the tangent to the curve at \( P \). The tangent to a curve at a point may, therefore, be defined as the limit to which a secant through the point tends as the other point of intersection with the curve tends to the given point. Since the two points of intersection of the secant and the curve tend to coincide, a tangent is sometimes defined as a straight line meeting the curve into ultimately coincident point.

1. (b) Geometrical Interpretation of the Differential Coefficient
Let \( P \) be any point \((x, y)\) on a given curve and \( Q \) be neighbouring point \((x + \delta x, y + \delta y)\) on the curve so that \( \delta x \) and \( \delta y \) are small quantities. Draw \( PL \) and \( QM \) perpendiculars on \( OX \) and \( PK \) perpendicular to \( QM \).

As is clear from the diagram
\[
\tan \angle QPK = \frac{QK}{PK} = \frac{\delta y}{\delta x},
\]
when
\[
Q \to P, \ \delta x \to 0,
\]
\[
\angle QPK \to \angle TPK = \Psi \ (\text{say})
\]
Proceeding to the limit as \( \delta x \to 0 \), we get
\[
\tan \Psi = \lim_{\delta x \to 0} \left( \frac{\delta y}{\delta x} \right) = \frac{dy}{dx} \ (\text{at the point } P).
\]
and equate the two results.
Thus the value of the differential coefficient at a point on the curve is the tangent of the angle which the tangent at that point makes with the X-axis \( i.e., \) the value of \( \frac{dy}{dx} \), at P is nothing but the slope of the tangent at P.

**Note.** If \( \frac{dy}{dx} \), at a point P on a curve is zero, the tangent to the curve at P is parallel to X-axis; conversely if the tangent at a point P is parallel to the X-axis, \( \frac{dy}{dx} \) is zero at P.

**Definition of Normal**

The straight line perpendicular to a tangent at its point of contact is called the normal to the curve at that point.

**Equation of the Tangent and Normal**

Let the equation of the curve be \( y = f(x) \) and let P \((x, y)\) be any point on it. Let the tangent PT at P to the curve meet the X-axis in M. Let \( \angle TMX = \Psi \), we have

\[
\tan \Psi = \frac{dy}{dx} \quad \text{(at) P} = f'(x).
\]  

\( \therefore \) Equation of the tangent PT is

\[
Y - y = \frac{dy}{dx} (X - x) = f'(x) (X - x) \quad \ldots \ (2)
\]

which is the line through \((x, y)\) having slope \( i.e. \ m = \frac{dy}{dx} \). Here \((X, Y)\) are the current coordinates \( i.e., \) the coordinates of any point on the line (1) and \((x, y)\) are the coordinates of P, where the tangent is drawn.
The normal PN being perpendicular to the tangent PT at P, its equation is given by

\[ Y - y = \frac{-1}{\frac{dy}{dx}}(X - x) \]

\[ i.e., \quad (X - x) + \frac{dy}{dx}(Y - y) = 0 \] \hspace{1cm} \ldots (2)

**Note 1.** In case \( \frac{dy}{dx} \) does not exist at \((x, y)\) but \( \frac{dx}{dy} = 0 \), then the tangent at \((x, y)\) can be obtained as follows:

In this case the tangent is parallel to the Y-axis and hence the equation of the tangent line is given by \( X = x \).

**Note 2.** When the equations of curve are given in parametric form viz.

\[ x = f(t), \quad y = g(t) \]

then

\[ \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)} \]

and the equations of the tangent and normal can be written immediately from (1) and (2) in the forms;

\[ Y - g(t) = \frac{g'(t)}{f'(t)}[X - f(t)] \]

and \( [X - f(t)] + \frac{g'(t)}{f'(t)}[Y - g(t)] = 0 \)

**Note 3.** The intercept of the tangent on X-axis is obtained by putting \( Y = 0 \) in (1).

\[ \therefore \text{Intercept on the X-axis is given by} \]

\[ X = x - \frac{y}{\frac{dy}{dx}} \]
and intercept of the tangent on Y-axis is obtained by putting X = 0 in (1).

∴ Intercept on the Y-axis is given by

\[ Y = y - x \frac{dy}{dx} \]

**Example 1.** Find the equations of tangent and normal to the parabola \( y^2 = 4ax \) at the point \((a, 2a)\).

**Solution.**

\[ y^2 = 4ax \]

Differentiating w.r.t. \(x\) we have,

\[ 2y \frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}, \quad y \neq 0. \]

∴ At the point \((a, 2a)\), \( \frac{dy}{dx} = \frac{2a}{2a} = 1 \).

∴ Tangent at \((a, 2a)\) is

\[ y - 2a = 1 (x - a), \]

\[ y = x + a \]

And the normal at \((a, 2a)\) is

\[ y - 2a = -(x - a) \]

or

\[ x + y = 3a. \]

**Example 2.** Find the tangent and the normal to the curve \( x = a (\theta - \sin \theta), \ y = a (1 - \cos \theta) \) at any point ‘\( \theta \)’.

**Solution.** Here

\[ \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a \sin \theta}{a (1 - \cos \theta)} = \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} = \cot \frac{\theta}{2}. \]

∴ Equation of tangent is:

\[ y - a (1 - \cos \theta) = \cot \frac{\theta}{2} \left[ x - a (\theta - \sin \theta) \right]. \]

Equation of normal is

\[ [y - a (1 - \cos \theta)] = \frac{-1}{\csc \frac{\theta}{2}} \left[ x - a (\theta - \sin \theta) \right] \]

or

\[ [y - a (1 - \cos \theta) \cdot \cot \frac{\theta}{2} + x - a (\theta - \sin \theta) = 0. \]

**Example 3.** Show that the line \( x \cos \theta + y \sin \theta = p \), will touch the curve \( x^m y^n = a^{m+n} \) if

\[ p^{m+n} m^n n^m = (m + n)^{m+n} a^{m+n} \cos^n \theta \sin^m \theta. \]

**Solution.** Let the line

\[ x \cos \theta + y \sin \theta = p, \quad \ldots \ (1) \]

be the tangent to the curve

\[ x^m y^n = a^{m+n}, \quad \ldots \ (2) \]

at the point \((\xi, \eta)\).
From (2), taking logarithms, we have
\[ m \log x + n \log y = (m + n) \log a. \]

Differentiating \( w.r.t. x \), we get
\[
\frac{m}{x} + \frac{n}{y} \frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{m}{n} \frac{y}{x} (y \neq 0)
\]

\[ \therefore \quad \left( \frac{dy}{dx} \right) \text{ at the point } (x_1, y_1) = -\frac{m y_1}{n x_1}. \]

Equation of tangent at \((x_1, y_1)\) to (2) is
\[ y - y_1 = -\frac{m y_1}{n x_1} (x - x_1) \]
or
\[ (m y_1) x + (n x_1) y = (m + n) x_1 y_1, \quad \ldots \text{ (3)} \]

If (1) is a tangent to (2) at \((x_1, y_1)\) then (1) and (3) must be identical.

\[ \therefore \text{ Comparing coefficients in (1) and (3), we get} \]
\[ \frac{m y_1}{\cos \theta} = \frac{n x_1}{\sin \theta} = \frac{(m + n)x_1 y_1}{p} \]

\[ \therefore \quad x_1 = \frac{pm}{(m + n)\cos \theta} \]

and
\[ y_1 = \frac{pn}{(m + n)\sin \theta} \]

Also since \((x_1, y_1)\) is a point on the curve (2), we get
\[ x_1^m y_1^n = a^{m+n}. \]

Substituting for \(x_1, y_1\), we have
\[ \frac{p^m m^m}{(m + n)^m \cos^m \theta} = \frac{p^m m^n}{(m + n)^n \sin^n \theta} = a^{m+n} \]
or
\[ p^{m+n} m^m n^n = (m + n)^{m+n} a^{m+n} \cos^m \theta \sin^n \theta \]

is the required condition.

**Example 4.** Find the equation of the tangent and normal to the curve \( y (x - 2) (x - 4) - x + 7 = 0 \) at the point where it cuts the X-axis.

**Solution.**
\[ y (x - 2) (x - 3) - x + 7 = 0 \]
\[ \Rightarrow \quad y (x^2 - 5x + 6) - x + 7 = 0 \quad \ldots \text{ (1)} \]

The curve (1) cuts the X-axis viz. \( y = 0 \)

where \( 0(x - 2) (x - 3) - x + 7 = 0 \)

or \( x = 7 \)

\[ \therefore \text{ The point of intersection is } (7, 0). \]
Differentiating both sides of (1) w.r.t. \( x \) we have:

\[
\frac{dy}{dx}(x - 2) (x - 3) + y (2x - 5) - 1 = 0
\]

At the point \((7, 0)\);

\[
\frac{dy}{dx}(7 - 2) (7 - 3) + 0 - 1 = 0.
\]

\[
\therefore \frac{dy}{dx} = \frac{1}{20}
\]

\[
\therefore \text{ Equation of tangent at the point } (7, 0) \text{ is}
\]

\[
y - 0 = \frac{1}{20} (x - 7) \quad \Rightarrow x - 20y - 7 = 0
\]

and equation of the normal at the point \((7, 0)\) is

\[
y - 0 = -20 (x - 7)
\]

\[
\Rightarrow 20x + y - 7 = 0.
\]

**Example 5.** Prove that the sum of the intercepts on the co-ordinate axes of any tangent to the curve \( \sqrt{x} + \sqrt{y} = \sqrt{a} \) is constant.

**Solution.** Equation of the curve is

\[
\sqrt{x} + \sqrt{y} = \sqrt{a}
\]

Differentiating both sides w.r.t. to \( x \), we get

\[
\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0
\]

\[
\Rightarrow \quad \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}
\]

\[
\therefore \text{ Equation of tangent to the given curve at } (x, y) \text{ is}
\]

\[
Y = y - \frac{\sqrt{y}}{\sqrt{x}} (X - x)
\]

\[
\Rightarrow \quad X\sqrt{y} + Y\sqrt{x} = x\sqrt{y} + y\sqrt{x}
\]

\[
\Rightarrow \quad X\sqrt{y} + Y\sqrt{x} = \sqrt{x} \sqrt{y} (\sqrt{x} + \sqrt{y})
\]

\[
\Rightarrow \quad X\sqrt{y} + Y\sqrt{x} = \sqrt{a} \sqrt{x} \sqrt{y} \quad \text{[} \because \sqrt{x} + \sqrt{y} = \sqrt{a} \text{]}
\]

Intercept on the axis of \( X \) is obtained by putting \( Y = 0 \)

\[
i.e., \quad \text{Intercept on } X\text{-axis} = \frac{\sqrt{a} \sqrt{x} \sqrt{y}}{\sqrt{y}} = \sqrt{\frac{ax}{y}}
\]
And Intercept on the axis of Y is obtained by putting X = 0 in the equation of tangent.

\[ \text{Intercept on y-axis} = \frac{\sqrt{ax} \sqrt{y}}{\sqrt{x}} = \sqrt{ay} \]

\[ \therefore \text{Sum of the intercepts on co-ordinate axes} = \sqrt{ax} + \sqrt{ay} . \]

\[ = \sqrt{a(\sqrt{x} + \sqrt{y})} \]

\[ = \sqrt{a} \sqrt{a} = a = \text{constant.} \]

\[ \therefore a \text{ is independent of } x \text{ and } y. \]

**Example 6.** Show that the length of the portion of the tangent to the curve \( x = a \cos^3 \theta, y = a \sin^3 \theta \) intercepted between the co-ordinate axes is constant.

**Solution.** We have

\[ \frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta. \]

\[ \frac{dy}{d\theta} = 3a \sin^2 \theta \cos \theta \]

\[ \therefore \frac{dy}{dx} = \frac{dy}{d\theta} \frac{d\theta}{dx} = \frac{-3a \sin^2 \theta \cos \theta}{3a \cos^2 \theta \sin \theta} = -\tan \theta \]

\[ \therefore \text{Equation of tangent at the point } \theta \text{' is} \]

\[ y - a \sin^3 \theta = -\tan \theta (x - a \cos^3 \theta) \]

\[ \Rightarrow x \sin \theta + y \cos \theta = a \sin \theta \cos \theta (\sin^2 \theta + \cos^2 \theta) \]

\[ \Rightarrow x \sin \theta + y \cos \theta = a \sin \theta \cos \theta \quad \ldots (1) \]

Tangent line (1) meets the co-ordinate axes at the points A \((a \cos \theta, 0)\) and B\((0, a \sin \theta)\). [These points are obtained by putting \( y = 0 \) and \( x = 0 \) in (1) respectively].

Thus the length of the portion of the tangent intercepted between the co-ordinate axes is given by

\[ AB = \sqrt{a^2 \cos^2 \theta + a^2 \sin^2 \theta} = a, \text{ which is constant.} \]

**Example 7.** Show that the tangent and normal at any point of the curve

\[ x = ae^\theta - (\sin \theta - \cos \theta) \]
\[ y = ae^\theta - (\sin \theta + \cos \theta) \]

are equidistant from the origin.

**Solution.** Here

\[ \frac{dx}{d\theta} = ae^\theta (\sin \theta - \cos \theta) + ae^\theta (\cos \theta + \sin \theta) \]

\[ = 2ae^\theta \sin \theta \]
and \[ \frac{dy}{d\theta} = ae^0 (\sin \theta + \cos \theta) + ae^0 (\cos \theta - \sin \theta) \]
\[ = 2ae^0 \cos \theta \]

\[ \therefore \frac{dy}{dx} = \frac{dy}{d\theta} \frac{d\theta}{dx} = \frac{2ae^0 \cos \theta}{2ae^0 \sin \theta} = \frac{\cos \theta}{\sin \theta} \]

\[ \therefore \text{Equation of the tangent to the given curve at any point } \theta \text{ is} \]
\[ y - ae^0 (\sin \theta + \cos \theta) = \frac{\cos \theta}{\sin \theta} \left[ x - ae^0 (\sin \theta - \cos \theta) \right] \]
\[ \Rightarrow x \cos \theta - y \sin \theta - ae^0 (\cos \theta \sin \theta - \cos^2 \theta - \sin^2 \theta - \sin \theta \cos \theta) = 0 \]
\[ \Rightarrow x \cos \theta - y \sin \theta + ae^0 = 0 \quad \ldots (1) \]

And equation of the normal at any point \( \theta \) to the given curve is
\[ y - ae^0 (\sin \theta + \cos \theta) = \frac{\sin \theta}{\cos \theta} \left[ x - ae^0 (\sin \theta - \cos \theta) \right] \]
\[ \Rightarrow x \sin \theta + y \cos \theta - ae^0 (\sin \theta \cos \theta + \cos^2 \theta + \sin^2 \theta - \sin \theta \cos \theta) = 0 \]
\[ \Rightarrow x \sin \theta + y \cos \theta - ae^0 = 0 \quad \ldots (2) \]

Now the tangent and the normal are equidistant from the origin if the length of the perpendicular from the origin up on the tangent (1) and normal (2) are equal. The length of the perpendicular from (0, 0) upon the line (1) is
\[ p_1 = \frac{0 \cos \theta - 0 \sin \theta + ae^0}{\sqrt{\cos^2 \theta + \sin^2 \theta}} = ae^0 \quad \ldots (3) \]

And the length of the perpendicular from (0, 0) upon the line (2) is
\[ p_2 = \frac{0 \sin \theta - 0 \cos \theta + ae^0}{\sqrt{\sin^2 \theta + \cos^2 \theta}} = -ae^0 \]
\[ = +ae^0 \quad \ldots (4) \]

[neglecting the \(-ve\) sign]

From (3) and (4), \( p_1 = p_2 \)

Hence the result.

**Example 8.** Show that tangent at any point of the curve
\[ x = a \ (t + \sin \ t \ \cos \ t) , \quad y = a \ (1 \ \sin \ t)^2 \]

makes an angle \( \frac{1}{4} (\pi + 2t) \) with the \( x = \) axis.

**Solution.** Here
\[ \frac{dx}{dt} = a \ [1 + \cos \ t \ \cos \ t + \sin \ t (- \sin \ t)] \]
\[ = a \ [1 + \cos^2 \ t - \sin^2 \ t] \]
\[ = a \, [1 + \cos 2t] = 2a \cos^2 t \]

and

\[ \frac{dy}{dt} = 2a \,(1 + \sin t) \cos t \]

\[ \therefore \quad \frac{dy}{dx} = \frac{2a(1 + \sin t) \cos t}{2a \cos^2 t} \]

\[ = \frac{1 + \sin t}{\cos t} = \frac{\left( \frac{\cos t}{2} + \frac{\sin t}{2} \right)}{\cos^2 \frac{t}{2} - \sin^2 \frac{t}{2}} \]

\[ = \frac{\cos \frac{t}{2} + \sin \frac{t}{2}}{\cos \frac{t}{2} - \sin \frac{t}{2}} \]

\[ = \frac{1 + \tan \frac{t}{2}}{1 - \tan \frac{t}{2}} = \tan \left( \frac{\pi}{4} + \frac{t}{2} \right) \]

If \( \theta \) is the angle which the tangent at any point \( Y' \) makes with the X-axis.

then

\[ \tan \theta = \tan \left( \frac{\pi}{4} + \frac{t}{2} \right) \]

\[ \therefore \quad \theta = \frac{\pi}{4} + \frac{t}{2} \]

\[ = \frac{1}{4}(\pi + 2t). \]

Hence the result.

**Example 9.** Prove that for the catenary \( y = c \cos \frac{x}{c} \), the perpendicular dropped from the foot of the ordinate upon the tangent is of constant length.

**Solution.** Hence

\[ y = c \cos \frac{x}{c} \]

\[ \Rightarrow \quad \frac{dy}{dx} = \frac{1}{c} \sinh \frac{x}{c} = \sinh \frac{x}{c} \]

Equation of tangent at any point \((x, y)\) is

\[ Y - y = \sinh \frac{x}{c}[X - x] \quad \text{[}\therefore \ y = c \cos \frac{x}{c}\text{]} \]
or \( Y - c \cosh \frac{x}{c} = \sinh \frac{x}{c} [X-x] \)

or \( X \left( \sinh \frac{x}{c} \right) - Y + c \cosh \frac{x}{c} - x \sinh \frac{x}{c} = 0 \)

The foot of the ordinate at the point \((x, y)\) is \((x, 0)\).

The length of the perpendicular from \((x, 0)\) on the tangent

\[
= \frac{x \sinh \frac{x}{c} - 0 + c \cosh \frac{x}{c} - x \sinh \frac{x}{c}}{\sqrt{1 + \sinh^2 \frac{x}{c}}}
\]

\[
= \frac{c \cosh \frac{x}{c}}{\cosh \frac{x}{c}}
\]

\[
= c, \text{ i.e., constant.}
\]

**Example 10.** Prove that \( \frac{x}{a} + \frac{y}{b} = 1 \) touches the curve \( y = be^{\frac{x}{a}} \) at the point where the curve crosses the y-axis.

**Solution.** \( y = be^{\frac{x}{a}} \) crosses y-axis i.e., the line \( x = 0 \)

where \( y = be^0 = b \)

\( \therefore \) The curve crosses y-axis at the point \((0, b)\).

Hence it is required to find the tangent at \((0, b)\).

Now

\[
\frac{dy}{dx} = \frac{be^{\frac{x}{a}}}{a}
\]

At \((0, b)\)

\[
\frac{dy}{dx} = -\frac{b}{a}
\]

\( \therefore \) Equation of tangent at \((0, b)\) is

\[
y - b = -\frac{b}{a} (x - 0)
\]

\( i.e., \)

\[
\frac{y}{b} - 1 = -\frac{x}{a}
\]

\( i.e., \)

\[
\frac{x}{a} + \frac{x}{b} = 1
\]

Hence proved.
Exercise

1. Find the equation of the tangent and normal to the curve \( x = a \cos \theta, y = b \sin \theta \) at any point '\( \theta \)'.

2. Find the tangent and normal to the curve
   \[
   \begin{align*}
   (i) & \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ at the point } (a, 0) \\
   (ii) & \quad x = a (\theta + \sin \theta), \quad y = a (1 + \cos \theta) \text{ at } \theta = \frac{\pi}{2}. \\
   (iii) & \quad y = c \cos h \frac{x}{c} \text{ at the point } (0, c)
   \end{align*}
   \]

3. Show that the line \( x \cos^3 \theta + y \sin^3 \theta = c \) is a tangent to the curve \( x^2 y^2 = a^2(x^2 + y^2) \).
   
   [Hint: The equation of the curve can be written as \( \frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{c^2} \).]

4. Prove that the straight line \( \frac{x}{a} + \frac{y}{b} = 2 \) touches the curve \( \left( \frac{x}{a} \right)^n + \left( \frac{y}{b} \right)^n = 2 \) at the point \( (a, b) \) whatever be the value of \( n \).

5. At what point of the curve \( y = x^2 - 3x + 2 \) is the tangent perpendicular to the line \( y = x \)?
   
   [Ans. \( (1, 0) \)]

6. At what point of the curve \( y = 2x^3 + 3x^2 - 10x + 7 \) are the tangents parallel to the line \( y = 2x \)?
   
   [Ans. \( (1, 2), (-2, 23) \)]

7. Prove that the equation of the tangent at any point (\( 4m^2 \), \( 8m^3 \)) of the semicubical parabola \( x^3 - y^2 = 0 \) is \( y = 3mx - 4m^3 \) and show that it meets the curve again at \( (m^2, -m^3) \), where it is normal if \( 9m^2 = 2 \).

8. Show that the normal at any point of the curve \( x = a \cos \theta + a \theta \sin \theta, y = a \sin \theta - a\theta \cos \theta \) is at a constant distance from the origin.

9. The tangent at any point on the curve \( x^3 + y^3 = 2a^3 \) cuts off lengths \( p \) and \( q \) on the co-ordinate axes, show that \( p^{3/2} + q^{-3/2} = 2^{1/2} a^{3/2} \).

Angle of Intersection of Two Curves

The angle at which two curves intersect at a point is defined as the angle between the tangents to the curves at that point. Applying the formula \( \tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2} \) for the angle \( \theta \) between the two lines whose gradients (slopes) are \( m_1 \) and \( m_2 \), the angle of intersection of two curves at a point of intersection is easily found.

When the angle between two curves at a point of intersection is a right angle, the curves are said to intersect orthogonally the condition for which is \( m_1 m_2 = -1 \).
Example 11. Find the angle of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4by$ at the point other than the origin.

Solution. The points of intersection of the parabolas $y^2 = 4ax$, and $x^2 = 4by$ are given by

$$x^4 = 16b^2y^2 = 16b^2 \cdot 4ax$$

or

$$x \ (x^3 - 64ab^2) = 0$$

$\Rightarrow$

$$x = 0 \quad \text{or} \quad x = 4a^{1/3} \ b^{2/3}$$

Substituting the value of $x$ in $x^2 = 4by$, we get

$$y = \frac{x^2}{4b} = \frac{16a^{2/3}b^{4/3}}{4b} = 4a^{2/3} \ b^{1/3}.$$  

:. (0, 0) and $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$ are the two points of intersection.

Now

$$y^2 = 4ax \quad \text{... (i)}$$

$$x^2 = 4by \quad \text{... (ii)}$$

Differentiating (i) w.r.t. $x$, we have

$$2y \ \frac{dy}{dx} = 4a \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y}$$

:. 

$$\left( \frac{dy}{dx} \right)_{(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})} = \frac{2a}{4a^{2/3}b^{1/3}} = \frac{a^{1/3}}{2b^{1/3}}$$

Differentiating (ii) w.r.t. $x$, we get,

$$2x = 4b \ \frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x}{2b}.$$
Thus if \( m_1, m_2 \) are the slopes of the tangents to the two curves, we have

\[
m_1 = \frac{a^{1/3}}{2b^{1/3}}, \quad m_2 = \frac{2a^{1/3}}{b^{1/3}}.
\]

\[\therefore\] The angle

\[
\theta = \tan^{-1}\left( \frac{m_2 - m_1}{1 + m_1m_2} \right)
\]

\[
= \tan^{-1}\left( \frac{2a^{1/3} - a^{1/3}}{2b^{1/3} + a^{1/3} - 2b^{1/3}} \right)
\]

\[
= \tan^{-1}\left( \frac{3a^{1/3}b^{1/3}}{2}\right).
\]

**Example 12.** Find the condition that the curves \( ax^2 + by^2 = 1 \) and \( a'x^2 + b'y^2 = 1 \) should cut orthogonally.

**Solution.** Let \((x_1, y_1)\) be the point of intersection. Then since the point lies on both the curves, we have

\[
ax_1^2 + by_1^2 - 1 = 0
\]

\[
a'x_1^2 + b'y_1^2 - 1 = 0
\]

\[\therefore\]

\[
\frac{x_1^2}{b' + b'} = \frac{y_1^2}{a' + a} = \frac{1}{ab' - a'b}
\]

\[
x_1^2 = \frac{b' - b}{ab' - a'b} \quad \text{and} \quad y_1^2 = \frac{a - a'}{ab' - a'b} \quad \ldots (1)
\]

Now differentiating \( ax^2 + by^2 = 1 \), we get

\[
2ax + 2by \frac{dy}{dx} = 0
\]

\[
\Rightarrow \quad \frac{dy}{dx} = \frac{-ax}{by}
\]

\[
\Rightarrow \quad m_1 = \left( \frac{dy}{dx} \right)_{(x_1, y_1)} = \frac{ax_1}{by_1}
\]

Similarly differentiating \( a'x^2 + b'y^2 = 1 \), we get
\[
2a'x + 2b'y \frac{dy}{dx} = 0
\]
\[\Rightarrow \quad \frac{dy}{dx} = -\frac{a'x}{b'y}
\]
\[\Rightarrow \quad m_2 = \left( \frac{dy}{dx} \right)_{(x_i, y_i)} = \frac{a'x_i}{b'y_i}
\]

The two curves will cut orthogonally if
\[m_1m_2 = -1
\]
i.e.,
\[\text{if } \left( \frac{a_i}{b_i} \right) \left( \frac{a'x_i}{b'y_i} \right) = -1
\]
\[\Rightarrow \quad aax_i^2 + bby_i^2 = 0
\]

Substituting the values of \( x_i^2 \) and \( y_i^2 \) from (1), we get,
\[ aa' \left( \frac{b' - b}{ab' - a'b} \right) + bb' \left( \frac{a - a'}{ab' - a'b} \right) = 0
\]
\[\Rightarrow \quad aa'(b' - b) + bb'(a - a') = 0
\]
\[\Rightarrow \quad \frac{b' - b}{bb'} + \frac{a - a'}{aa'} = 0
\]
\[\Rightarrow \quad \frac{b' - b}{bb'} = \frac{a' - a}{aa'}
\]
\[\Rightarrow \quad \frac{1}{b'} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{a}
\]

which is the required condition.

**Exercises**

Find the angle between the following pairs of curves at each one of their points of intersection

(i) \( x^2 - y^2 = a^2, \ x^2 + y^2 = \sqrt{2} a^2 \)

(ii) Prove that the curves \( y = 1 - ax^2 \) and \( y = x^2 \) cut orthogonally when \( a = \frac{1}{3} \).

(iii) Find the angle of intersection of the following curves:
\[
x^2 + 2xy - y^2 + 2ax = 0
\]
\[
3y^3 - 2a^2x - 4x^2y + a^3 = 0
\]
at the point \((a, -a)\).
Lengths of the Tangent, the Normal, the Sub-tangent and the Sub-normal

Let the tangents at any point \( P(x, y) \) on a curve meet the \( X \)-axis in \( T \), then \( PT \) is the length of the tangent intercepted between the point \( P \) and the \( X \)-axis and is often called the length of the tangent. The projection of this length on \( OX \) is called the subtangent. Here \( TM \) is the subtangent (where \( PM \) is the perpendicular on \( OX \)).

If the normal at \( P \) to the curve meets \( OX \) in \( N \), then \( PN \) is called the length of the normal and its projection \( MN \) on \( OX \) is called the subnormal.

These lengths are easily obtained from the right angled triangles \( PTM \) and \( PMN \).

We have

\[
\angle PTM = \angle MPN = \Psi ,
\]

where

\[
\tan \Psi = \left( \frac{dy}{dx} \right) \text{ at } P
\]

\[
\therefore \text{ PT, the length of the tangent at } P = PM \csc \Psi
\]

\[
\therefore \frac{PM}{PT} = \sin \Psi \Rightarrow PT = PM \csc \Psi \text{ and } PM = y
\]

\[
= y\sqrt{1 + \cot^2 \Psi}
\]

\[
= y\sqrt{1 + \tan^2 \Psi} = y\sqrt{1 + \left( \frac{dy}{dx} \right)^2}
\]

\[
= y \frac{dx}{dy} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} = y \sqrt{1 + \left( \frac{dx}{dy} \right)^2}
\]
TM, the length of the subtangent

\[\text{TM} \cot \Psi = \frac{\text{PM}}{\tan \Psi} = \frac{y}{x} = y \frac{dx}{dy} \quad \text{[\because \text{PM} = \tan \Psi]} \]

Also PN, the length of the normal

\[\text{PN} \sec \Psi = y \sqrt{1 + \tan^2 \Psi} \quad \text{[\because \frac{\text{PM}}{\text{PN}} = \cos \Psi \therefore \text{PN} = \text{PM} \sec \Psi]} \]

\[= y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \]

MN, the length of the subnormal

\[\text{PM} \tan \Psi = y \frac{dy}{dx} \quad \text{[\because \text{PM} = \tan \Psi]} \]

**Example 13.** Show that the subtangent at any point of the exponential curve \(y = ae^{x/b}\) is constant and the subnormal varies as the square of the ordinate.

**Solution.** Here

\[\frac{dy}{dx} = a \frac{1}{b} e^{x/b} = \frac{y}{b} \]

\[\therefore \text{Subtangent at } (x, y) = y \frac{dx}{dy} = y \frac{b}{y} = b \]

which is a constant since \(b\) is independent of \(x, y\).

Subnormal = \(y \frac{dy}{dx} = \frac{y^3}{b}\) which varies as the square of the ordinate \(y\).

**Example 14.** Show that for the curve \(by^2 = (x + a)^3\), the square of subtangent varies as subnormal.

**Solution.** Equation of the curve is

\[by^2 = (x + a)^3\]

Differentiating w.r.t. \(x\), we get

\[2by \frac{dy}{dx} = 3(x + a)^2 \]

\[\Rightarrow \frac{dy}{dx} = \frac{3(x + a)^2}{2by} \]

Length of subtangent

\[= \frac{y}{dy} = \frac{y}{3(x + a)^3} = \frac{2by^2}{3(x + a)^2} \quad \text{[\because y^2 = \frac{(x + a)^3}{b}]} \]

[18]
\[ = \frac{2}{3} (x + a) = T \quad \text{(say)} \]

Length of subnormal
\[ = \frac{dy}{dx} = y \cdot \frac{3(x+a)^2}{2by} = \frac{3}{2b} (x + a)^2 = N \quad \text{(say)} \]

\[ \Rightarrow \quad (x + a)^2 = \frac{2bN}{3} \]

Square of subtangent \[ = \frac{4}{9} (x + a)^2 \]

\[ \Rightarrow \quad T^2 = \frac{4}{9} \cdot \frac{2bN}{3} = \left(\frac{8b}{27}\right)N = kN \quad \text{[where } k = \frac{8b}{27} \text{ = constant]} \]

\[ \therefore T^2 \propto N. \text{ Hence proved.} \]

**Example 15.** Show that in the curve \( y = a \log (x^2 - a^2) \) the sum of the tangent and the subtangent varies as the product of the co-ordinates of the point.

**Solution.** Here \( y = a \log (x^2 - a^2) \)

\[ \therefore \frac{dy}{dx} = a \cdot \frac{2x}{x^2 - a^2} = \frac{2ax}{x^2 - a^2} \]

\[ \therefore \text{Length of Subtangent} \quad = y \frac{dx}{dy} = \frac{y(x^2 - a^2)}{2ax} \]

And length of tangent
\[ = y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \]
\[ = y \sqrt{1 + \left(\frac{x^2 - a^2}{2ax}\right)^2} \]
\[ = y \sqrt{\frac{4a^2x^2 + (x^2 - a^2)}{4a^2x^2}} \]
\[ = \frac{y}{2ax} \sqrt{(x^2 + a^2)^2} = \frac{y(x^2 + a^2)}{2ax} \]

Hence the sum of the tangent and the subtangent
\[ = \frac{y(x^2 - a^2)}{2ax} + \frac{y(x^2 + a^2)}{2ax} = \frac{y}{2ax} (x^2 - a^2 + x^2 + a^2) = \frac{y2x^2}{2ax} \]
\[ = \frac{xy}{a} = \frac{1}{a} (xy) \text{ which varies as } xy. \]
Exercises

1. Find the lengths of the subtangent, subnormal, tangent and normal for the following curves
   
   (i) \(2x^2 - 3y^2 = 15\) at \((3, 1)\)
   
   (ii) \(x = a \cos^3 \theta, y = a \sin^3 \theta\) at \(\theta\).
   
   (iii) \(x = a (\theta - \sin \theta), y = a (1 - \cos \theta)\) at \(\theta = \frac{\pi}{2}\).

2. Find the lengths of the normal and subnormal to the curve
   
   \[y = \frac{a}{2} \left[ e^{x/a} + e^{-x/a} \right].\]

3. Show that the subtangent at any point of the curve \(x^m y^n = a^{m+n}\) varies as the abscissa.

4. Show that in the parabolas \(y^2 = 4ax\), the subnormal is constant and the subtangent varies as the abscissa of the point of contact.

5. Prove that in the ellipse \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\), the length of the normal varies inversely as the perpendicular from the origin upon the tangent.

6. For the catenary \(y = c \cos \left( \frac{x}{c} \right)\), prove that the length of normal is \(\frac{y^2}{c}\).

7. Show that the subnormal at any point of curve \(y^2 x^2 = a^2 (x^2 - a^2)\) varies inversely as the cube of the abscissa.

8. Show that for the curve \(by^2 = (x + a)^3\), the square of the subtangent varies as the subnormal.

9. Show that in the curve \(y = a \log (x^2 - a^2)\), the sum of the tangent and the subtangent varies as the product of the co-ordinates of the point.
LESSON 2

TANGENTS AND NORMALS (POLAR COORDINATES)

Angle between Radius Vector and Tangent

Let P be a given point \((r, \theta)\) on the curve \(r = f(\theta)\) and Q a neighbouring point on the curve very close to P whose coordinates are \((r + \delta r, \theta + \delta \theta)\) so that \(r + \delta r = f(\theta + \delta \theta)\).

Let the tangent PT at P makes \(\angle TPM\) equal to \(\phi\) with the radius sector OP. To find \(\phi\) we observe that PT is the limiting position of the secant PQ when \(Q \to P\).

\[\therefore \quad \angle TPM = \lim_{Q \to P} \angle QPM\]

From Q, draw QM ⊥ on OP. Then

\[\tan \angle QPM = \frac{QM}{PM} = \frac{(r + \delta r)\sin \delta \theta}{(r + \delta r)\cos \delta \theta - r} \quad \text{[} \therefore \quad PM = OM - OP = OM - r\text{]}

Now when \(Q \to P\), then \(\delta \theta \to 0\) and \(\delta r \to 0\)

\[\therefore \quad \tan \delta = \lim_{Q \to P} \tan \angle QPM

= \lim_{\delta \theta \to 0} \frac{(r + \delta r)\sin \delta \theta}{(r + \delta r)\cos \delta \theta - r}

= \lim_{\delta \theta \to 0} \frac{(r + \delta r)\sin \delta \theta}{r \left( \frac{\cos \delta \theta - 1}{\delta \theta} \right) + \frac{\delta r}{\delta \theta} \cos \delta \theta}

= \frac{(r + \delta r)\sin \delta \theta}{r \frac{\cos \delta \theta - 1}{\delta \theta} + \frac{\delta r}{\delta \theta} \cos \delta \theta} \quad \text{[} \therefore \quad PM = OM - OP = OM - r\text{]}

\]
But we know that
\[
\lim_{\delta \theta \to 0} \left( \frac{\sin \delta \theta}{\delta \theta} \right) = 1
\]

Also
\[
\frac{1 - \cos \delta \theta}{\delta \theta} = \frac{2 \sin^2 \frac{\delta \theta}{2}}{\delta \theta}
\]

\[
= \left( \frac{\sin \frac{\delta \theta}{2}}{\frac{\delta \theta}{2}} \right) \sin \frac{\delta \theta}{2}
\]

\[
\therefore \lim_{\delta \theta \to 0} \left( \frac{1 - \cos \delta \theta}{\delta \theta} \right) = \lim_{\delta \theta \to 0} \left( \frac{\sin \frac{\delta \theta}{2}}{\frac{\delta \theta}{2}} \right) \lim_{\delta \theta \to 0} \left( \sin \frac{\delta \theta}{2} \right)
\]

\[
= 1 \times 0 = 0
\]

\[
\therefore \tan \phi = \lim_{\delta \theta \to 0} \frac{r}{dr} = r \frac{d\theta}{dr}
\]

Then angle \( \phi \) is the angle which the positive direction of the tangent \( i.e., \) the direction \( n \) which \( \theta \) increases makes with the positive direction of the radius vector \( i.e., \) the direction in which \( r \) increases. This angle \( \phi \) lies between 0 and \( \pi \) and when \( \phi \) is obtuse, \( \tan \phi \) is negative.

**Polar Subtangent and Polar Subnormal**

Let O be the origin and OX the initial line. Let P \( (r, \theta) \) be any point on the curve whose polar equation is given and let the perpendicular at O to the radius vector OP meet the tangent at P and T
and normal at P in N. Then OT is called the polar subtangent at P and ON the polar subnormal at P. Also PT is the length of tangent and PN the length of the normal at P.

From the right angled ΔOPT, we have polar subtangent = OT = OP tan ϕ.

From ΔOPN, we have

Polar subnormal = ON = OP cot ϕ

Length of tangent = PT = OP sec ϕ

And length of normal = PN = OP cosec ϕ

Note: A negative value of polar subtangent or the polar subnormal implies that in the curve at the point under consideration r decreases as θ increases.

**Angle of Intersection in Polar Coordinates**

The angle of intersection of two curves is evidently the difference between the values of ϕ for them at the point of intersection. If these angles are ϕ₁, ϕ₂, the required angle is given by

\[ \tan(ϕ₁ - ϕ₂) = \frac{\tan ϕ₁ - \tan ϕ₂}{1 + \tan ϕ₁ \tan ϕ₂} \]

where

\[ \tan ϕ₁ = \left( \frac{dϕ}{dr} \right) \] for the first curve

\[ \tan ϕ₂ = \left( \frac{dϕ}{dr} \right) \] for the second curve

when \( ϕ₁ - ϕ₂ = \frac{π}{2} \), the curves are said to intersect orthogonally.

i.e., when \( \tan ϕ₁ \tan ϕ₂ = -1 \), the two curves intersect orthogonally.

\[ ϕ₁ = ϕ₂ \]
i.e., when \( \phi_1 - \phi_2 = 0 \), the two curves touch.

i.e., when \( \tan \phi_1 = \tan \phi_2 \) the two curves touch.

**Example 1.** For the cardioid \( r = a \ (1 - \cos \theta) \) prove that

(i) \( \phi = \frac{1}{2} \theta \)

(ii) Polar subtangent \( = 2a \sin^2 \frac{\theta}{2} \tan \frac{\theta}{2} \).

**Solution.** Here \( \frac{dr}{d\theta} = a \sin \theta \)

\[ \tan \phi = r \frac{d\theta}{dr} = \frac{a(1 - \cos \theta)}{a \sin \theta} \]

\[ = \frac{2\sin^2 \frac{\theta}{2}}{2\sin \frac{\theta}{2} \cos \frac{\theta}{2}} \]

\[ = \tan \frac{\theta}{2} \]

\[ \therefore \phi = \frac{1}{2} \theta \]

Polar subtangent \( = r^2 \frac{d\theta}{dr} \)

\[ = r \left( r \frac{d\theta}{dr} \right) \]

\[ = a \ (1 - \cos \theta), \ \tan \phi \]

\[ = a \ (1 - \cos \theta) \cdot \tan \frac{\theta}{2} \]

\[ \therefore \phi = \frac{1}{2} \theta \]

**Example 2.** Find the angle of intersection of the two curve \( r = a \cos \theta \) and \( r = a \ (1 - \cos \theta) \).

**Solution.** The two curves \( r = a \cos \theta \) and \( r = a \ (1 - \cos \theta) \) intersect when

\[ \cos \theta = 1 - \cos \theta \]

\[ \Rightarrow \quad 2 \cos \theta = 1 \]

\[ \Rightarrow \quad \cos \theta = \frac{1}{2} \]
\[
\theta = 2n\pi + \frac{\pi}{3}, \text{ where } n \text{ is any integer}
\]

Let us take the value \[\theta = \frac{\pi}{3}.\]

For the curve \[r = a \cos \theta, \quad \frac{dr}{d\theta} = -a \sin \theta\]

\[\therefore \tan \phi_1 = \left( \frac{dr}{d\theta} \right)_2 = \left( \frac{a \cos \theta}{-a \sin \theta} \right) = -\cot \theta\]

Also for the curve \[r = a \left(1 - \cos \theta\right), \quad \frac{dr}{d\theta} = a \sin \theta\]

\[\tan \phi_2 = \left( \frac{dr}{d\theta} \right)_2 = \frac{1 - \cos \theta}{\sin \theta}\]

\[= \frac{2 \sin^2 \theta}{2 \sin \theta \cos \theta} \frac{\theta}{2 \cos \theta} \frac{\theta}{2}\]

\[= \tan \frac{\theta}{2}\]

\[\tan (\phi_1 - \phi_2) = \frac{\tan \phi_1 - \tan \phi_2}{1 + \tan \phi_1 \tan \phi_2}\]

\[= \frac{-\cot \theta - \tan \frac{\theta}{2}}{1 - \cot \theta \cdot \tan \frac{\theta}{2}}\]

The curves intersect, when \[\theta = \frac{\pi}{3}\]

\[\tan (\phi_1 - \phi_2) = \frac{-\cot \frac{\pi}{3} - \tan \frac{\pi}{6}}{1 - \cot \frac{\pi}{3} \cdot \tan \frac{\pi}{6}}\]

\[= \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3} \cdot \sqrt{3}}\]
\[
\begin{align*}
\frac{-2}{\frac{2\sqrt{3}}{3}} &= -\sqrt{3} \\
\phi_1 - \phi_2 &= \tan^{-1}(\sqrt{3}) \\
&= \frac{\pi - \frac{\pi}{3}}{3} = \frac{2\pi}{3}
\end{align*}
\]

\therefore The required angle is \(\frac{2\pi}{3}\).

**Example 3.** Show that the curves \(r = a(1 + \cos \theta)\) and \(r = b(1 - \cos \theta)\) intersect orthogonally.

**Solution.** The two curves will intersect orthogonally when \(\phi_1 - \phi_2 = \frac{\pi}{2}\) the condition for which is \(\tan \phi_1 \cdot \tan \phi_2 = -1\).

or \(1 + \tan \phi_1 \tan \phi_2 = 0\)

For the first curve \(r = a(1 + \cos \theta)\) we have

\[
\left(\frac{dr}{d\theta}\right) = -a \sin \theta
\]

\therefore \(\tan \phi_1 = \left(\frac{r}{dr}\right) = \frac{1 + \cos \theta}{\sin \theta}\)

Similarly for the second curve \(r = b(1 - \cos \theta)\)

\[
\left(\frac{dr}{d\theta}\right) = b \sin \theta
\]

\[
\tan \phi_2 = \left(\frac{r}{dr}\right) = \frac{1 - \cos \theta}{\sin \theta}
\]

\therefore \(\tan \phi_1 \tan \phi_2 = -\left(\frac{1 + \cos \theta}{\sin \theta}\right)\left(\frac{1 - \cos \theta}{\sin \theta}\right)\)

\[
= -\frac{(1 - \cos^2 \theta)}{\sin^2 \theta}
\]

\[
= -1 \quad (\because \theta \neq 0)
\]

\[\Rightarrow\] The curves intersect orthogonally.
Exercises

1. Find the angle \( \phi \) for the curve

   \[
   (i) \quad \frac{2a}{r} = 1 - \cos \phi \\
   (ii) \quad r^m = a^m \cos m\theta
   \]

2. Find the angle of intersection of curves \( r = \sin \theta + \cos \theta \) and \( r = 2 \sin 2\theta \).

3. Show that in the equiangular spiral \( r = ae^{\theta \cos \alpha} \), the tangent is inclined at a constant angle to the radius vector.

   [Hint: Prove that \( \phi = \alpha \)]

   Also show that the polar subtangent is \( r \tan \alpha \) and polar subnormal is \( r \cot \alpha \).

4. Show that in the curve \( r = a\theta \), the subnormal is constant and in the curve \( r\theta = a \), the polar subtangent is constant.

5. For the cardioid \( r = a(1 - \cos \theta) \), prove that

   \[
   (i) \quad \text{Polar subtangent} = 2r \sin^2 \frac{\theta}{2} \tan \frac{\theta}{2} \\
   (ii) \quad \text{Polar tangent} = 2r \sin^2 \frac{\theta}{2} \sec \frac{\theta}{2} \\
   (iii) \quad \text{Polar normal} = 2r \sin \frac{\theta}{2} \\
   (iv) \quad \text{Polar subnormal} = a \sin \theta.
   \]

6. Show that the logarithmic spiral \( r = ae^{b\theta} \) has the lengths of its polar tangent, polar normal, polar subtangent and polar subnormal each proportional to \( r \).

7. Prove that the two curves \( r = \frac{a}{1-\cos \theta} \) and \( r = \frac{a}{1+\cos \theta} \) cut orthogonally.

The Perpendicular from the Pole on the Tangent

Let \( P \) be a point on a curve, distant \( r \) from the pole. Let \( ON \) be the perpendicular from \( O \) on the tangent at \( P \). Then the length of \( ON \) denoted by \( p \) is an important quantity and can sometimes be used as a coordinate to define the position of \( P \). Evidently from the right angled \( \triangle ONP \), we have,
\[ p = r \sin \phi \] \hspace{1cm} \ldots \hspace{0.5cm} (1)

Also we have
\[ \tan \phi = r \frac{d\theta}{dr} = r \frac{dr}{d\theta} \]

\[ \therefore \quad p = r \sin \phi = -\frac{r^2}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}} \]

\[ \therefore \quad \frac{1}{p^2} = \frac{r^2 + \left(\frac{dr}{d\theta}\right)^2}{r^2} \]

\[ = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta}\right)^2 \]

A neat form of this is obtained if in place of \( r \), we used
\[ u = \frac{1}{r}, \]

\[ \therefore \quad \frac{du}{d\theta} = -\frac{1}{r^2} \frac{dr}{d\theta} \]

\[ \therefore \quad \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \]

\textbf{Pedal Equation}

The relation between the quantities \( p \) and \( r \) of any point on a curve is called the pedal equation of the curve. For some curves this equation is very simple.

To obtain the pedal equation of a curve whose equation is given in cartesian form we use the formula
\[ r^2 = x^2 + y^2 \] \hspace{1cm} \ldots \hspace{0.5cm} (1)

The length of the perpendicular from the origin on the tangent
\[ Y - y = \frac{dy}{dx} (X - x) \] is given by
\[ p = \frac{x \frac{dy}{dx} - y}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \] \hspace{1cm} \ldots \hspace{0.5cm} (2)
[\text{i.e.,} \text{Length of } \perp \text{ from } (0, 0) \text{ to the straight line } ax + by + c = 0 \text{ is } \frac{c}{\sqrt{a^2 + b^2}}]

The value of \( \frac{dy}{dx} \) is obtained from the equation of the curve viz.

\[ f(x, y) = 0 \]  \hspace{1cm} \text{... (3)}

Eliminating \( x, y \) from (1), (2) and (3) the equation obtained is called “Pedal equation”.

\textbf{Note:} When the position of the pole is not mention then it is to be taken at the origin of the cartesian axes of coordinates.

To obtain the pedal equation from the polar equation

\[ r = f(\theta) \text{ or } f(r, \theta) = 0 \] \hspace{1cm} \text{... (4)}

We use the relation

\[ \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left( \frac{dr}{d\theta} \right)^2 \] \hspace{1cm} \text{... (5)}

The elimination of \( \theta' \) between (4) and (5) gives the pedal equation.

\textbf{Example 4.} Show that the pedal equation of the parabola \( y^2 = 4a(x + a) \) is \( p^2 = ar \).

\textbf{Solution.} Here \( y^2 = 4a(x + a) \)

\[ 2y \frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y} \]

\[ \therefore \text{Equation of tangent at any point } (x, y) \text{ is} \]

\[ Y - y = \frac{2a}{y} (X - x) \]

or \[ 2aX - yY + y^2 - 2ax = 0. \]

The length of the perpendicular \( p' \) from the origin on the tangent is given by

\[ p = \frac{y^2 - 2ax}{\sqrt{4a^2 + y^2}} \]

\[ = \frac{y^2 - 2ax}{\sqrt{4a^2 + 4a(x + a)}} \hspace{1cm} [\therefore y^2 = 4a(x + a)] \]

\[ = \frac{4a(x + a) - 2ax}{\sqrt{8a^2 + 4ax}} \]

\[ = \frac{4ax + 4a^2 - 2ax}{\sqrt{4a(2a + x)}} \]

\[ = \frac{2a(x + 2a)}{2\sqrt{a(x + 2a)}} = \sqrt{2a(x + a)} \]
Also\[ r^2 = x^2 + y^2 = x^2 + 4a(x + a) = (x + 2a)^2 \]
\[ \Rightarrow \]
\[ r = x + 2a \]
\[ \therefore \]
\[ p^2 = a(x + 2a) = ar \]
\[ p^2 = ar \] is the required pedal equation.

**Example 5.** Show that the pedal equation of the curve \[ r^2 = a^2 \cos \Theta \] \[ r^3 = a^2 p. \]

**Solution.** \[ r^2 = a^2 \cos \Theta \]

Differentiating w.r.t. \( \Theta \) we have:

\[ 2 \frac{dr}{d\Theta} = -2a^2 \sin 2\Theta \]

\[ \therefore \]

\[ \frac{dr}{d\Theta} = -\frac{a^2 \sin 2\Theta}{r} \]

Also

\[ \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left( \frac{dr}{d\Theta} \right)^2 \]

\[ = \frac{1}{r^2} + \frac{1}{r^4} \left( \frac{a^2 \sin 2\Theta}{r} \right)^2 \]

\[ = \frac{1}{r^2} + \frac{1}{r^6} a^4 \sin^2 2\Theta \]

From (1), \( \cos \Theta = \frac{r^2}{a^2} \)

\[ \therefore \]

\[ \sin 2\Theta = \sqrt{1-\cos^2 2\Theta} = \sqrt{1-\frac{r^4}{a^4}} \]

\[ \therefore \]

\[ \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^6} a^4 \left( \frac{r^4}{a^4} - 1 \right) \]

\[ = \frac{1}{r^2} + \frac{a^4}{r^6} - \frac{1}{r^2} \]

\[ = \frac{a^4}{r^6} \]

\[ \therefore \]

\[ p^2 = \frac{r^6}{a^4} \]

\[ \Rightarrow \]

\[ p = \frac{r^3}{a^2} \]

\[ \therefore \]

\( r^3 = a^2 p \) is the required pedal equation.
Exercise

1. Obtain the pedal equation of the following curves
   (i) \( r = a (1 - \cos \theta) \)
   (ii) \( \frac{2a}{r} = 1 - \cos \theta \)
   (iii) \( r^n = a^n \sin n\theta \)
   (iv) \( r = ae^\theta \cot \alpha \)
   (v) \( r^2 \cos 2\theta = a^2 \)

2. For the parabola \( \frac{2a}{r} = 1 - \cos \theta \), show that polar subtangent is \( 2a \csc \theta \) and \( p = a \csc \frac{\theta}{2} \).

Derivative of Arc

Let the tangents at two neighbouring points P and Q on a curve meet in T, then we shall assume that

\[
\lim_{Q \to P} \frac{\text{arcPQ}}{\text{chord PQ}} = 1.
\]

Derivative of Arc in Cartesian Coordinates

Let P be any point \((x, y)\) on a curve and Q a neighbouring point \((x + \delta x, y + \delta y)\) on the curve, very close to P. Draw PM \(\perp\) from P on the ordinate QL.

\[
\text{Chord PQ} = \sqrt{PM^2 + QM^2} = \sqrt{(\delta x)^2 + (\delta y)^2} \quad \ldots \quad (1)
\]

Let A be any fixed point on the curve of which PQ is an arc. Let the length of the arc AP = \(s\) and that of arc AQ be \(s + \delta s\) so that arc PQ = \(\delta s\).
Then from (1),
\[
\frac{\delta s}{\delta x} = \frac{\delta s}{\operatorname{chord PQ}} \cdot \frac{\operatorname{chord PQ}}{\delta x} = \frac{\operatorname{Arc PQ}}{\operatorname{chord PQ}} \cdot \sqrt{\frac{\left(\frac{\delta x}{\delta x}\right)^2 + \left(\frac{\delta y}{\delta x}\right)^2}{x}}
\]
\[
= \frac{\operatorname{Arc PQ}}{\operatorname{chord PQ}} \cdot \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}
\]

When Q→P, then \(\delta x \to 0\) and \(\lim_{Q \to P} \frac{\operatorname{arc PQ}}{\operatorname{chord PQ}} = 1\)

∴ Taking limits as \(\delta x \to 0\), we get
\[
\frac{dx}{dx} = \lim_{\delta x \to 0} \frac{\delta x}{\delta x} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}
\]

The positive value of the radical being taken if the convention is made that 'y' is so measured as to make it increase with \(x\) increasing, as in the figure.

Multiplying both sides of (2) by \(\frac{dx}{dy}\), we have
\[
\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}
\]
which is useful when the equation of the curve is given in the form of \(x = f(y)\).

When the equation of a curve is given in parametric form \(x = f(t), y = g(t)\)

then
\[
\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}
\]

**Corollary:** Putting \(\frac{dy}{dx} = \tan \psi\), where \(\psi\) is the angle which the tangent makes with the X-axis we have
\[
\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \tan^2 \psi} = \sec \psi
\]
∴
\[
\cos \psi = \frac{dx}{ds}
\]

Also
\[
\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \cot^2 \psi} = \csc \psi
\]
∴
\[
\frac{dy}{ds} = \sin \psi.
\]
Derivatives of the Arc in Polar Coordinates

Let P and Q be two neighbouring points on a curve where actual distances from a fixed point A say on the curve are \(s\) and \(s + \delta s\) respectively. Let the polar coordinates of P and Q be \((r, \theta)\) and \((r + \delta r, \theta + \delta \theta)\).

Join PQ. Through P draw PM perpendicular to OQ. Then from right angled \(\Delta PQM\).

\[
\text{PQ}^2 = \text{PM}^2 + \text{MQ}^2
\]

\[
= (r \sin \delta \theta)^2 + [r + \delta r - r \cos \delta \theta]
\]

\[
\therefore \quad \text{MQ} = \text{OQ} - \text{OM} = (r + \delta r) - \text{OP} \cos \delta \theta
\]

\[
\therefore \quad \left( \frac{\text{PQ}}{\delta \theta} \right)^2 = r^2 \left( \frac{\sin \delta \theta}{\delta \theta} \right)^2 + \left[ \frac{\delta r + 2r \sin \delta \theta}{(\delta \theta)^2} \right]
\]

\[
\Rightarrow \quad \left( \frac{\text{PQ}}{\delta \theta} \right)^2 = r^2 \left( \frac{\sin \delta \theta}{\delta \theta} \right)^2 + \left[ \frac{\delta r}{(\delta \theta)} + 2r \left( \frac{\sin \delta \theta}{2} \right) \frac{1}{\delta \theta} \right]^2
\]

\[
\therefore \quad 1 - \cos \delta \theta = 2 \sin \frac{\delta \theta}{2}
\]

Now

\[
\lim_{\delta \theta \to 0} \left( \frac{\text{PQ}}{\delta \theta} \right) = \lim_{\delta \theta \to 0} \left( \frac{\text{PQ}}{\text{arc PQ}} \cdot \frac{\text{arc PQ}}{\delta \theta} \right)
\]

\[
= \frac{ds}{d\theta}
\]

\[
\therefore \quad \lim_{\delta \theta \to 0} \left( \frac{\text{chord PQ}}{\text{arc PQ}} \right) = 1
\]

Also

\[
\lim_{\delta \theta \to 0} \left( \frac{\sin \delta \theta}{\delta \theta} \right) = 1, \quad \lim_{\delta \theta \to 0} \left( \frac{\delta r}{\delta \theta} \right) = \frac{dr}{d\theta}
\]
\[
\lim_{\delta \theta \to 0} \frac{\sin \frac{\delta \theta}{2}}{\delta \theta} \left( \frac{\sin \frac{\delta \theta}{2}}{\frac{\delta \theta}{2}} \right) = 0
\]

\[
\therefore \text{On taking limits as } \delta \theta \to 0, \text{ we get}
\]

\[
\left( \frac{ds}{d\theta} \right)^2 = r^2 + \left( \frac{dr}{d\theta} \right)^2
\]

\[
\Rightarrow \quad \frac{ds}{d\theta} = \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2}
\]

If the equation of the curve is \( \theta = f(r) \),

then it can be shown that

\[
\frac{ds}{dr} = \sqrt{1 + r^2 \left( \frac{d\theta}{dr} \right)^2}
\]

**Other Formulae**

Since

\[
r \frac{d\theta}{dr} = \tan \phi
\]

\[
\Rightarrow \quad \frac{dr}{d\theta} = r \cot \phi
\]

\[
\therefore \quad \left( \frac{ds}{d\theta} \right)^2 = r^2 + \left( \frac{dr}{d\theta} \right)^2 = r^2 + r^2 \cot^2 \phi
\]

\[
= r^2 (1 + \cot^2 \phi)
\]

\[
= r^2 \cosec^2 \phi
\]

\[
\Rightarrow \quad \frac{ds}{d\theta} = r \cosec \phi
\]

\[
\therefore \quad \sin \phi = \frac{r}{ds} \frac{d\theta}{ds}
\]

the positive root being taken if the length of the arc is measured in the direction in which \( \theta \) increases.

We have

\[
\sin \phi = \frac{r}{ds} \frac{d\theta}{ds} = \frac{r}{dr} \frac{dr}{ds}
\]
\[ \tan \phi = \frac{dr}{ds} \quad \text{[}\because \tan \phi = r \frac{d\theta}{dr}\text{]} \]

\[ \therefore \quad \frac{dr}{ds} = \frac{\sin \phi}{\tan \phi} = \cos \phi \]

\[ \Rightarrow \quad \cos \phi = \frac{dr}{ds} \]

**Example 6.** Prove that for curve \( r = a e^{\theta \cot \alpha} \), \( d \) is constant, \( s \) being measured from the origin.

**Solution.** Equation of the given curve is

\[ r = a e^{\theta \cot \alpha} \]

\[ \therefore \quad \log r = \log a + 0 \cot \alpha \]

Differentiating w.r.t. \( \theta \) we get,

\[ \frac{1}{r} \frac{dr}{d\theta} = \cot \alpha \]

\[ \therefore \quad r \frac{d\theta}{dr} = \tan \alpha \]

\[ i.e., \quad \tan \phi = \tan \alpha \quad \Rightarrow \quad \phi = \alpha \]

\[ \therefore \quad \frac{dr}{ds} = \cos \phi = \cos \alpha \]

or

\[ \frac{ds}{dr} = \sec \alpha \]

Integrating, we get

\[ s = r \sec \alpha + c \]

Assuming that when \( r = 0 \), we get \( c = 0 \)

\[ \therefore \quad s = r \sec \alpha \quad \Rightarrow \quad \frac{s}{r} = \sec \alpha = \text{constant}. \]

**Example 7.** Show that for the curve

\[ \theta = \cos^{-1} \frac{r}{k} \frac{-\sqrt{k^2 - r^2}}{r}, \]

\[ = \frac{r \frac{ds}{dr}}{dr} \quad \text{is constant}. \]

**Solution.** Differentiating the given equation w.r.t. \( r \) we get,

\[ \frac{d\theta}{dr} = \frac{-1}{\sqrt{\frac{r^2}{k^2} - \frac{k^2 - r^2}{r^2}}} \cdot \left( -\frac{r}{k^2} \frac{r - \sqrt{k^2 - r^2}}{r^2} \right) \]
\[
\frac{dr}{d\theta} = \frac{1}{\frac{d\theta}{dr}} = \frac{r^2}{\sqrt{k^2 - r^2}} \quad \text{(where } r \neq \pm k)\
\]

Also
\[
\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2
\]
\[
= r^2 + \frac{r^4}{k^2 - r^2} = \frac{k^2 r^2}{k^2 - r^2}
\]
\[
\Rightarrow \quad \frac{ds}{d\theta} = \frac{kr}{\sqrt{k^2 - r^2}}
\]
\[
\frac{r}{dr} \frac{ds}{d\theta} = r \frac{ds}{d\theta} \cdot \frac{d\theta}{dr}
\]
\[
= r \left(\frac{kr}{\sqrt{k^2 - r^2}}\right) \frac{\sqrt{k^2 - r^2}}{r^2} \quad \text{(where } r \neq \pm k)\]
\[
= k, \text{ which is constant.}
\]

**Example 8.** Show that for a curve, whose pedal equation is \( p = f(r) \)

\[
\frac{ds}{dr} = \frac{r}{\sqrt{r^2 - p^2}}
\]

**Solution.** For any curve,

\[
\cos \phi = \frac{dr}{ds},
\]
\[
p = r \sin \phi,
\]
\[
\text{... (i)}
\]
we have

\[
\frac{ds}{dr} = \frac{1}{\cos \phi}
\quad \ldots \ (ii)
\]

\[
= \frac{r}{r \cos \phi}
\quad \text{where } r \neq 0
\]

\[
= \frac{r}{\sqrt{r^2 \cos^2 \phi}}
\]

\[
= \frac{r}{\sqrt{r^2 (1 - \sin^2 \phi)}}
\]

\[
= \frac{r}{\sqrt{r^2 - p^2}}
\quad [\therefore \ p = r \sin \phi]
\]

**Exercises**

1. Find \( \frac{ds}{dx} \frac{ds}{dy} \) for the following curves.

   (i) \( y = c \cos h \frac{x}{c} \)

   (ii) \( x^3 = ay^2 \).

2. Find \( \frac{ds}{d\theta} \) for the following curves

   (i) \( x = a \cos \theta, y = b \sin \theta \)

   (ii) \( x = a (\theta - \sin \theta), y = a (1 - \cos \theta) \)

   (iii) \( r^2 = a^2 \cos \theta \).

3. Show that for any curve \( \frac{ds}{d\theta} = \frac{r^2}{p} \).
1. Definitions

Consider a point moving on a small arc PQ of a curve. The direction of motion at any point being along the tangent to the curve at the point, it is clear from the figure that the direction of motion changes from KPT to KQT while the point describes the arc PQ. Thus the direction turns through an angle TMT’ i.e., TMQ when the moving point traverses the arc PQ. A measure of the rate of turning (i.e., the rate of change in direction) is \( \frac{\angle TMQ}{\text{arc PQ}} \).

This measure gives us an idea of the average curvature of the arc PQ. We shall now obtain precise measure of curvature at a point P on a curve.

Let A be a fixed point on the curve such that arc AP = \( s \) and the arc AQ = \( s + \delta s \), so that the length of the arc PQ = \( \delta s \). Let \( \psi \) be the angle which the tangent at P makes with the positive direction of the \( x \)-axis.

\[ \angle TKX = \psi. \]

Let the angle which the tangent K' MQT' at Q makes with positive direction of the \( x \)-axis be \( \psi + \delta \psi \) where \( \delta \psi \) is a small angle depending on the curve and the position of Q relative to P on it. The angle through which the direction of the tangent changes when arc \( \delta s \) is traversed is therefore equal to \( \angle TMT \). An approximate measure of the curvature of the arc PQ is, therefore \( \frac{\delta \psi}{\delta s} \).

Now let Q→P, then \( \delta s \to 0 \) and \( \delta \psi \to 0 \), and we have
\[
\frac{\delta \psi}{\delta s} = \frac{Lt}{\delta_{s \to 0}} \frac{\delta \psi}{\delta s} = \text{curvature at P}
\]

The reciprocal of curvature is called the *radius of curvature* and is generally denoted by the greek letter $\rho$. Thus $\rho = \frac{ds}{d\psi}$.

**2. Curvature of a Circle**

Let PQ be an arc of a circle whose centre is C. If $\angle PCQ = \theta$, i.e., the angle subtended at the centre of the circle by the arc PQ of the circle) and the radius $CP = r$, then arc $PQ = r\theta$ ($\theta$ being measured in radians).

The angle between the tangents at P and Q is also $\theta$ since $\angle CPT = 90^\circ = \angle CQT$.

Hence the curvature $\frac{1}{\rho}$ of the arc PQ is $\frac{\theta}{r\theta}$ i.e., $\frac{1}{r}$ which is independent of the magnitude of $\theta$. Thus the radius of curvature $\rho$ at a point on the circle is $r$ (the radius of the circle).

We also see that in a circle the normals at any two points P and Q meet at the point C called the centre of curvature and length of the normal at any point P up to the centre of curvature, is the radius of curvature at P.

**3. Cartesian Formulae for $\rho$**

When the equation of a curve is given in rectangular (cartesian) co-ordinates viz $y = f(x)$, then radius of curvature is obtained as follows:

We know that $\frac{dy}{dx} = \tan \psi$ and $\frac{ds}{dx} = \sec \psi$

enable us to express $\frac{ds}{d\psi}$, the formula for $\rho$, in terms of $\frac{dy}{dx}$ and its derivatives. We have

\[
y_1 = \frac{dy}{dx} = \tan \psi
\]
Differentiating with respect to $x$, we get

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}(\tan \psi) = \frac{d}{d\psi}(\tan \psi) \frac{d\psi}{dx}$$

$$= \sec^2\psi \frac{d\psi}{dx} = \sec^2\psi \frac{d\psi}{ds} \cdot \frac{ds}{dx}$$

But

$$\frac{d\psi}{ds} = \frac{1}{\rho} \quad \text{and} \quad \frac{ds}{dx} = \sec \psi$$

$\therefore$

$$y^2 = \sec^2\psi \cdot \frac{1}{\rho} \cdot \sec \psi = \frac{1}{\rho} \sec^3 \psi$$

$$= \frac{1}{\rho} (1 + \tan^2 \psi)^{3/2}$$

$$= \frac{1}{\rho} (1 + y_1^2)^{3/2}$$

$\therefore$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} \quad \ldots \quad (1)$$

**Note 1.** The radius of curvature of a curve is an intrinsic property of the curve. It depends on the curve itself and not on the choice of the axes of co-ordinates. The relation between $s$ and $\psi$ obtained by integrating $\frac{ds}{d\psi}$, is called the *intrinsic equation of the curve.*

**Note 2.** If at any point of curve the tangent is parallel to the $y$-axis, $\frac{dy}{dx}$ does not exist there, and the formula (1) above becomes meaningless. In this case $\frac{dx}{dy} = 0$, at the point under consideration and we can employ derivatives of $x$ w.r.t. $y$. For example; starting with the differentiation of the relation

$$\frac{dx}{dy} = \cot \psi$$

with respect to $y$ we get, as before,

$$\therefore \quad \rho = \frac{1 + (\frac{dx}{dy})^2}{{ds}^2} \quad \ldots \quad (2)$$

**Note 3.** Generally $s$ and $\psi$ are measured that one increases with the other, so that the derivative $\frac{ds}{d\psi}$, i.e., $\rho$ is positive. Accordingly while extracting the square root involved in (1) and (2), that sign
is taken in the formulae (1) and (2) which gives a positive value to $\rho$.

**Solved Example 1.** Prove that the radius of curvature at any point $(x, y)$ on the catenary $y = c \cos h \frac{x}{c}$ is $\frac{y^2}{c}$.

**Solution.** We have 

$$y = \cos \frac{x}{c},$$

so that

$$y_1 = c \sinh \frac{x}{c},$$

and

$$y_2 = \frac{1}{c} \cosh \frac{x}{c}.$$

Therefore

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{\left[1 + \sinh^2 \frac{x}{c}\right]^{3/2}}{\frac{1}{c} \cosh \frac{x}{c}}$$

and consequently

$$\rho = c \cos \frac{x}{c} = \frac{y^2}{c}.$$  

**Example 2.** Show that for any curve

(i) $\frac{1}{\rho} = \frac{d}{dx} \left(\frac{dy}{ds}\right)$

(ii) $\frac{1}{\rho} = \frac{y_2}{(1 + y_1^2)^{3/2}}$

**Solution.** (i) We have

$$\frac{d}{dx} \left(\frac{dy}{ds}\right) = \frac{d}{dx} \left(\frac{dy}{dx} \frac{dx}{ds}\right)$$

and

$$\frac{d}{dx} \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{d}{dx} \left(\frac{y_1}{\sqrt{1 + y_1^2}}\right).$$
Solved Example 3. Show that for any curve \( \frac{1}{\rho} = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}} \) when the equation of a curve is given in parametric form viz., \( x = f(t), y = g(t) \) where accents denote differentiation w.r.t. \( t \) i.e., \( x' = \frac{dx}{dt} \) and \( y' = \frac{dy}{dt} \).

Solution. In the formula
\[
\frac{1}{\rho} = \frac{y_2}{(1 + y_1^2)^{3/2}} \quad \cdots (1)
\]
We have to express \( y_1 \) and \( y_2 \) in terms of the derivatives of \( x \) and \( y \) w.r.t. \( t \). We know that
\[
\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{y'}{x'}
\]
i.e.,
\[
y_1 = \frac{y'}{x'}
\]
Hence
\[
y_2 = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{dy_1}{dx}
\]
\[
= \frac{dt}{dx} \left( \frac{\frac{y'}{x'}}{x'} \right)
\]
\[
= \frac{1}{x'} \left( \frac{y'' x' - x' y''}{(x')} \right)
\]
\[
= \frac{y'' x' x' y'}{(x')}^3
\]
Thus

\[
\frac{1}{\rho} = \frac{y''x' - x''y'}{(x')^3} \sqrt{1 + \left(\frac{y'}{x'}\right)^{3/2}}
\]

\[
= \frac{y''x' - x''y'}{(x')^3 + (x')^{3/2}}
\]

**Solved Example 4.** In the cycloid

\[x = a(t + \sin t), \quad y = a(1 - \cos t),\]

Prove that

\[\rho = 4a \cos \frac{t}{2}\]

**Solution.**

\[x = a(t + \sin t), \quad y = a(1 - \cos t)\]

\[\therefore \quad \frac{dx}{dt} = a(1 + \cos t)\]

\[\frac{dy}{dt} = a \sin t\]

\[\frac{d^2x}{dt^2} = -a \sin t,\]

\[\frac{d^2y}{dt^2} = a \cos t.\]

Now from the solved example 2, we have

\[\frac{1}{\rho} = \frac{x'y'' - y''x'}{(x'^2 + y'^2)^{3/2}}\]

where

\[x' = \frac{dx}{dt}, \quad y' = \frac{dy}{dt}, \quad x'' = \frac{d^2x}{dt^2}, \quad y'' = \frac{d^2y}{dt^2}\]

\[\therefore \quad \frac{1}{\rho} = \frac{a(1 + \cos t) \cos t - a \sin t (-\sin t)}{\left[a^2(1 + \cos t)^2 + a^2 \sin^2 t\right]^{3/2}}\]

\[i.e., \quad \frac{1}{\rho} = \frac{a^2\left[\cos t + \cos^2 t + \sin^2 t\right]}{a^4\left[1 + \cos^2 t + 2 \cos t + \sin^2 t\right]^{3/2}}\]

\[= \frac{\left[\cos t + 1\right]}{a\left[2 + \cos t\right]^{3/2}}\]

\[= \frac{\left[1 + \cos t\right]}{a2^{3/2}(1 + \cos t)\sqrt{1 + \cos t}}\]
\[
\rho = 4a \cos \frac{t}{2}
\]

We can also find the value of \( \rho \) by using the formulae

\[
\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}
\]

Now

\[
y_1 = \frac{dy}{dx} = \frac{dy}{dt} = \frac{a \sin t}{a(1 + \cos t)}
\]

\[
= \frac{2 \sin \frac{t}{2} \cos \frac{t}{2}}{2 \cos^2 \frac{t}{2}} = \tan \frac{t}{2}
\]

\[
y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx} \left( \tan \frac{t}{2} \right) = \frac{1}{2} \sec^2 \frac{t}{2} \frac{dt}{dx}
\]

\[
= \frac{1}{2} \sec^2 \frac{t}{2} \cdot \frac{1}{2} \frac{dx}{dt} = \frac{1}{2} \sec^2 \frac{t}{2}
\]

\[
= \frac{\sec^2 \frac{t}{2}}{2 a \cdot 2 \cos^2 \frac{t}{2}} = \frac{1}{4 a \cos^2 \frac{t}{2}}
\]

\[
\therefore \quad \rho = \frac{(1 + \tan^2 t)^{3/2}}{1} = 4a \cos \frac{t}{2} \sec^3 \frac{t}{2}
\]

\[
\rho = 4a \cos \frac{t}{2}.
\]
Solved Example 5. If CP, CD be a pair of conjugate semi diameters of the ellipse
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \]
prove that the radius of curvature at P is \( \frac{(CD)^3}{ab} \).

**Solution.** We know that P \((a \cos \theta, b \sin \theta)\) and D \((-a \sin \theta, b \cos \theta)\) are the extremities of CP and CD.

Equation of the ellipse is
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]
\[ \Rightarrow \quad b^2x^2 + a^2y^2 = a^2b^2 \]
Differentiating with respect to \(x\), we get
\[ 2b^2x + 2a^2yy_1 = 0 \]
\[ \Rightarrow \quad y_1 = -\frac{b^2x}{a^2y} = -\frac{b^2}{a^2} \left( \frac{x}{y} \right) \]
\[ y_2 = -\frac{b^2}{a^2} \frac{y - xy_1}{y_2} = -\frac{b^2}{a^2} \left( \frac{y + \frac{b^2 x^2}{a^2 y}}{y^2} \right) \]
[By putting the value of \(y_1\)]
\[ = -\frac{b^2}{a^2} \left[ \frac{a^2 y^2 + b^2 x^2}{a^2 y^2} \right] = -\frac{b^2}{a^2} \left( \frac{1 + \frac{b^2 x^2}{a^2 y^2}}{a^2 y^2} \right) = -\frac{b^4}{a^4 y^2} \]

Now
\[ \rho = \left( \frac{1 + y_1^2}{|y_2|} \right)^{3/2} = \left( \frac{1 + \frac{b^4 x^2}{a^4 y^2}}{b^4 a^4 y^2} \right)^{3/2} = \left( \frac{a^4 y^2 + b^4 x^2}{a^4 b^4} \right)^{3/2} \]

At P \((a \cos \theta, b \sin \theta)\),
\[ \rho = \frac{(a^4 b^2 \sin^2 \theta + b^4 a^2 \cos^2 \theta)^{3/2}}{a^4 b^4} \]
\[ = \frac{a^4 b^4 (a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{a^4 b^4} \]
\[ = \frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{ab} \]
Also \[ CD = \sqrt{(0+a\sin\theta)^2+(0-b\cos\theta)^2} = \sqrt{a^2\sin^2\theta+b^2\cos^2\theta} \]

Hence \[
\rho = \frac{a^2\sin^2\theta+b^2\cos^2\theta} {ab} \bigg(\frac{CD}{ab}\bigg)^{\frac{3}{2}}.
\]

**Solved Example 6.** Prove that for the ellipse \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \] \( \rho = \frac{a^2b^2}{p^2} \), \( \rho \) being the perpendicular from the centre upon the tangent at any point \((x, y)\).

**Solution.** From solved Example 5,

\[ \rho = \frac{\left(a^4y^2+b^4x^2\right)^{\frac{3}{2}}}{a^4b^4} \quad \ldots \ (1) \]

The equation of the tangent to the given ellipse at \((x, y)\) is

\[ Y - y = -\frac{b^2}{a^2} \cdot \frac{x}{y} (X-x) \]

\[ \Rightarrow \frac{Xx}{a^2} + \frac{Yy}{b^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \]

\[ \Rightarrow \frac{Xx}{a^2} + \frac{Yy}{b^2} - 1 = 0 \]

\( \therefore \) The perpendicular distance from the centre upon the tangent is

\[ p = \frac{1}{\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}} = \frac{a^2b^2}{\sqrt{a^4y^2 + b^4x^2}} \]

\[ \Rightarrow \sqrt{a^4y^2 + b^4x^2} = \frac{a^2b^2}{p} \]

Substituting this value in (1), we get

\[ \rho = \frac{a^2b^2}{p^2} \cdot \frac{1}{a^4b^4} = \frac{a^2b^2}{p^3} \quad \text{Proved.} \]

**Exercises–I**

1. Find the radius of curvature at any point \((x, y)\) on the following curves
   (i) \( y^2 = 4ax \)
   (ii) \( ay^2 = x^3 \)
   (iii) \( xy = c^2 \)

2. Show that the radius of curvature at the point \((a \cos^3\theta, a \sin^3\theta)\) on the curve \( x^{2/3} + y^{2/3} = a^{2/3} \) is \( 3a \sin \theta \cos \theta \).
[**Hint:** The parametric equations of the curve are

\[ x = a \cos^3 \theta \]
\[ y = a \sin^3 \theta \]

Use the formulas obtained in solved example 2]

3. Find the radius of curvature at the specified point on the following curves

   (i) \( \sqrt{x} + \sqrt{y} = 1 \), at the point \( \left( \frac{1}{4}, \frac{1}{4} \right) \).

   (ii) \( y = 4 \sin x - \sin 2x \), at the point \( x = \frac{\pi}{2} \).

   (iii) \( x^3 + y^3 = 3axy \) at the point \( \left( \frac{3a}{2}, \frac{3a}{4} \right) \).

4. \( \rho \) For Pedal Equations

   The relation between \( \rho \) and \( r \) for points on a curve is called the *pedal equation* of the curve. We shall now obtain an expression for \( \rho \) suitable for curves given by pedal equations.

   From the diagram it is evident that

   \[ \psi = \theta + \phi \]  
   \[ r \frac{d\theta}{ds} = \sin \phi \]  
   \[ \frac{dr}{ds} = \cos \phi \]  
   \[ p = r \sin \phi \]  
   \[ \rho = \frac{ds}{d\psi} \]

   and

   From the diagram it is evident that

   \[ \psi = \theta + \phi \]  
   \[ r \frac{d\theta}{ds} = \sin \phi \]  
   \[ \frac{dr}{ds} = \cos \phi \]  
   \[ p = r \sin \phi \]  
   \[ \rho = \frac{ds}{d\psi} \]
We shall now eliminate $\Theta, \phi, s$ and $\psi$ from these relations and obtain the value of $\rho$ in terms of $p$ and $r$.

\[
\frac{1}{\rho} = \frac{d\psi}{ds}
\]

\[
= \frac{d}{ds}(\Theta + \phi) = \frac{d\Theta}{ds} + \frac{d\phi}{ds}
\]

\[
= \frac{\sin \phi}{r} + \frac{d\phi}{dr} \cdot \frac{dr}{ds} = \frac{\sin \phi}{r} + \frac{d\phi}{dr} \cdot \cos \phi
\]

\[
= \frac{p}{r} \cdot \frac{1}{r} + \frac{d}{dr}(\sin \phi) = \frac{p}{r^2} + \frac{d}{dr} \left(\frac{p}{r}\right)
\]

\[
= \frac{p}{r^2} + \left[ -\frac{p}{r^2} + \frac{1}{r} \cdot \frac{dp}{dr} \right] = \frac{1}{r} \cdot \frac{dp}{dr}
\]

\[
\therefore \quad \rho = r \frac{dr}{dp}
\]

**Solved Example 7.** Find the radius of curvature at the point ($p$, $r$) for the ellipse

\[
\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}
\]

**Solution.** Differentiating the given equation w.r.t. $p$, we get

\[
-\frac{2}{p^3} = -\frac{2r}{a^2b^2} \cdot \frac{dr}{dp},
\]

i.e.,

\[
\frac{dr}{dp} = \frac{a^2b^2}{rp}
\]

\[
\therefore \quad \rho = r \frac{dr}{dp} = \frac{a^2b^2}{p}
\]

**Exercise—II**

1. Find the radius of curvature at any point on the following curves:

   (i) $p^2 = ar$ (parabola)

   (ii) $pr = a^2$ (hyperbola)

   (iii) $r^3 = 2ap^2$ (cardioid)

   (iv) $r^3 = a^2p$ (lemniscate)

2. In the curve $p = \frac{r^{n+1}}{a^n}$ show that the radius of curvature varies inversely as the $(n - 1)$th power of the radius vector.
3. Prove that for the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), \( \rho = \frac{a^2b^2}{p^2} \) where \( p \) is the perpendicular from the centre upon the tangent at \((x, y)\).

5. Polar Formula For \( \rho \)

To obtain the formula for the radius of curvature when the equation of a curve is given in polar coordinates we make use of the following results

\[
\frac{dr}{ds} = \cos \phi, \psi = \theta + \phi, \quad r \frac{d\theta}{dr} = \tan \phi
\]

we have

\[
\frac{d\psi}{ds} = \frac{d\psi}{d\theta} \frac{d\theta}{ds} \quad \text{... (1)}
\]

Also,

\[
\frac{d\psi}{d\theta} = 1 + \frac{d\phi}{d\theta} \quad \text{... (2)}
\]

\[
\frac{ds}{d\theta} = \sqrt{r^2 \left( \frac{dr}{d\theta} \right)^2}
\]

\[
\tan \phi = r \frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}}
\]

Differentiating w.r.t. \( \theta \), we have

\[
\sec^2 \phi \cdot \frac{d\phi}{d\theta} = \frac{\left( \frac{dr}{d\theta} \right)^2 - r \frac{d^2r}{d\theta^2}}{\left( \frac{dr}{d\theta} \right)^2}
\]

\[
\because \quad \frac{d\phi}{d\theta} = \frac{\left( \frac{dr}{d\theta} \right)^2 - r \frac{d^2r}{d\theta^2}}{1 + \frac{r^2}{\left( \frac{dr}{d\theta} \right)^2}} \quad (\because \sec^2 \phi = 1 + \tan^2 \phi)
\]

\[
= \frac{\left( \frac{dr}{d\theta} \right)^2 - r \frac{d^2r}{d\theta^2}}{r^2 + \left( \frac{dr}{d\theta} \right)^2}
\]

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Thus from (2), we get
\[
\frac{d\psi}{d\theta} = 1 + \frac{d\phi}{d\theta}
\]
\[
= 1 + \frac{\left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2 r}{d\theta^2}}{r^2 + \left(\frac{dr}{d\theta}\right)^2}
\]
\[\therefore\] Putting the values of \(\frac{d\psi}{d\theta}\) and \(\frac{ds}{d\theta}\) in (1), we obtain
\[
\frac{d\psi}{ds} = \frac{1 + \left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2 r}{d\theta^2}}{r^2 + \left(\frac{dr}{d\theta}\right)^2}^{3/2}
\]
\[
= \frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2 r}{d\theta^2}}{r^2 + \left(\frac{dr}{d\theta}\right)^2}^{3/2}
\]
\[\therefore\] \(\frac{1}{\rho} = \frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2 r}{d\theta^2}}{r^2 + \left(\frac{dr}{d\theta}\right)^2}^{3/2}\) \((\because \rho = d)\)
\[
\rho = \frac{r^2 + \left(\frac{dr}{d\theta}\right)^2}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2 r}{d\theta^2}}^{3/2}
\]
If we write \(\frac{dr}{d\theta} = r_1, \frac{d^2 r}{d\theta^2} = r_2\) the formula becomes
\[
\rho = \frac{\left(r^2 + r_1^2\right)^{3/2}}{r^2 + 2r_1^2 - rr_2}.
\]
Solved Example 8. Show that for the cardioid

\[ r = a \ (1 + \cos \theta), \quad \rho = \frac{4a}{3} \cos \frac{\theta}{2}. \]

Also prove that \( \frac{\rho^2}{r} \) is constant.

Solution. Here

\[ r_1 = \frac{dr}{d\theta} = -a \sin \theta, \quad r_2 = -a \cos \theta. \]

\[ \therefore \quad \rho = \frac{(r^2 + r_1^2)^{3/2}}{(r^2 + 2r_1^2 - r_2^2)} \]

\[ = \frac{a^2 (1 + \cos \theta)^2 + a^2 \sin^2 \theta}{{a^2 (1 + \cos \theta)^2 + 2a^2 \sin^2 \theta + a^2 \cos \theta (1 + \cos \theta)}}^{3/2} \]

\[ = \frac{a^3 [1 + \cos^2 \theta + 2\cos \theta + a^2 \sin^2 \theta]}{a^2 [1 + \cos^2 \theta + 2\cos \theta + 2\sin^2 \theta + \cos \theta + \cos^2 \theta]} \]

\[ = \frac{a (2 + 2\cos \theta)^{3/2}}{3 + 3\cos \theta} = \frac{2\sqrt{2a}}{3} \sqrt{1 + \cos \theta} \]

\[ = \frac{2}{3} \sqrt{2a} \sqrt{\frac{\cos^2 \theta}{2}} = \frac{4a \cos \theta}{3} \]

Squaring, we get \( \rho^2 = \frac{16a^2 \cos^2 \theta}{9} = \frac{8a^2}{9} (1 + \cos \theta) = \frac{8a}{9} r \)

\[ \therefore \quad \frac{\rho^2}{r} = \text{constant}. \]

Solved Example 9. For the curve \( r^m = a^m \cos m\theta \), prove that

\[ \rho = \frac{a^m}{(m+1)r^{m-1}} \]

Solution.

\[ r^m = a^m \cos m\theta \]

\[ \therefore \quad m \log r = m \log a + \log \cos m\theta \]

Differentiating both sides with respect to \( \theta \), we get

\[ \frac{m}{r} \frac{dr}{d\theta} = -m \frac{\sin m\theta}{\cos m\theta} \]
\[ r_1 = -\tan m\theta \]
\[ r_2 = -r_1 \tan m\theta - r \sec^2 m\theta \cdot m \]
\[ = r \tan^2 m\theta - mr \sec^2 m\theta \quad [\because r_1 = -r \tan m\theta] \]
\[ \rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2} \]
\[ = \frac{\left[ r^2 + r^2 \tan^2 m\theta \right]^{3/2}}{r^2 + 2r^2 \tan^2 m\theta - r^2 \tan^2 m\theta + mr^2 \sec^2 m\theta} \]
\[ = \frac{r^3 \sec^3 m\theta}{r^2 \sec^2 m\theta + mr^2 \sec^2 m\theta} = \frac{r \sec m\theta}{(m + 1)} \]
\[ = \frac{r}{(m+1)\cos m\theta} \frac{a^m}{(m + 1)r^m} = \frac{a^m}{(m + 1)r^{m-1}} \]

**Solved Example 10.** If \( \rho_1 \) and \( \rho_2 \) be the radii of curvature at the extremities of any chord of the cardioid \( r = a (1 + \cos \theta) \) which passes through the pole, then \( \rho_1^2 + \rho_2^2 = \frac{16a^2}{9} \).

**Solution.** For \( r = a (1 + \cos \theta) \)
\[ r_1 = \frac{dr}{d\theta} = -a \sin \theta \]
\[ r_2 = \frac{d^2r}{d\theta^2} = -a \cos \theta \]

Now
\[ \rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2} \]
\[ = \frac{a^3 \left(1 + \cos \theta\right)^2 + a^2 \sin^2 \theta}{a^2 \left(1 + \cos \theta\right)^2 + 2a^2 \sin^2 \theta - a(1 + \cos \theta)(-a \cos \theta)} \]
\[ = \frac{a^3 \left[1 + 2\cos \theta + \cos^2 \theta + \sin^2 \theta\right]^{3/2}}{a^2 \left[1 + 2\cos \theta + \cos^2 \theta + 2\sin^2 \theta + \cos \theta + \cos^2 \theta\right]} \]
\[ = \frac{a^3 \left[2(1 + \cos \theta)\right]^{3/2}}{3 + 3\cos \theta} \]
\[ = \frac{2\sqrt{2}a(1 + \cos \theta)^{2/3}}{3(1 + \cos \theta)} \]
Let $PO$ be any chord of the curve passing through the pole. If the vectorial angle at $P$ is $\theta$, then vectorial angle at $Q$ is $\pi + \theta$.

Now

$$\rho = \frac{4a}{3} \cos \frac{\theta}{2}$$

At $P$,

$$\rho_1 = \frac{4a}{3} \cos \frac{\theta}{2}$$

and at $Q$,

$$\rho_2 = \frac{4a}{3} \cos \left( \frac{\pi + \theta}{2} \right)$$

$$= -\frac{4a}{3} \sin \frac{\theta}{2}$$

Hence

$$\rho_1^2 + \rho_2^2 = \frac{16}{9} \cos^2 \frac{\theta}{2} + \frac{16}{9} \sin^2 \frac{\theta}{2}$$

$$= \frac{16}{9} a^2.$$ 

**Solved Example 11.** Show that in the curve \(r^2 = a^2 \sin 2\theta\), the tangent turns three times as fast as the radius vector and that the curvature varies as the radius vector.

**Solution.**

\[ r^2 = a^2 \sin 2\theta \quad \ldots \quad (i) \]

From (i)

\[ 2r \frac{dr}{d\theta} = 2a^2 \cos 2\theta \]

\[ \therefore \quad \frac{d\theta}{dr} = \frac{r}{a^2 \cos 2\theta} \quad \ldots \quad (ii) \]

\[ \therefore \quad \tan \phi = \frac{r}{a^2 \cos 2\theta} = \tan 2\theta \quad (\because r^2 = a^2 \sin 2\theta) \]

\[ \Rightarrow \quad \phi = 2\theta \]
Also \( \psi = \theta + \phi \)
\[ \psi = \theta + \Theta = \Theta \]

\[ \frac{d\psi}{ds} = 3 \frac{d\theta}{ds}, \] which shows that the tangent turns three times as fast as the radius vector.

\[ \frac{d\psi}{ds} = 3 \frac{d\theta}{ds} \quad \Rightarrow \quad \frac{ds}{d\psi} = \frac{1}{3} \frac{ds}{d\theta} \]

We have
\begin{align*}
\frac{ds}{d\theta} &= \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \\
&= \sqrt{r^2 + a^4 \cos^2 2\theta} \\
&= \frac{1}{r} \sqrt{a^4 \sin^2 2\theta + a^4 \cos^2 2\theta} \\
&= \frac{a^2}{r}
\end{align*}

\[ \therefore \quad \frac{ds}{d\psi} = \frac{1}{3} \frac{ds}{d\theta} = \frac{a^2}{3r} \]
or
\[ \frac{d\psi}{ds} = 3 \frac{r}{a^2} = kr, \]
where
\[ k = \frac{3}{a^2} = \text{constant} \]

\textit{i.e., curvature varies as the radius vector.}

6. Radius of Curvature for Tangential Polar Equations

A relation between \( p \) and \( \psi \) on a curve is called tangential polar equation. We make use of the following known formulae

\[ \rho = r \frac{dr}{dp}, \quad \rho = \frac{ds}{d\psi}, \quad p = r \sin \phi \]

and
\[ \frac{dr}{ds} = \cos \phi \]

The two expressions for \( \rho \) when equated give
\[ r \frac{dr}{dp} = \frac{ds}{d\psi} \]
\[
\frac{dp}{d\psi} = r \frac{dr}{ds} = r \cos \phi
\]

Eliminating \( \psi \) using \( p = r \sin \phi \), we have

\[
p^2 + \left( \frac{dp}{d\psi} \right)^2 = r^2 \left( \sin^2 \phi + \cos^2 \phi \right) = r^2.
\]

Also

\[
\rho = r \frac{dr}{dp} = \frac{1}{2} \frac{d}{dp} \left( r^2 \right)
\]

\[
= \frac{1}{2} \frac{d}{dp} \left[ p^2 + \left( \frac{dp}{d\psi} \right)^2 \right]
\]

\[
= \frac{1}{2} \cdot 2p + \frac{1}{2} \frac{d}{d\psi} \left[ \left( \frac{dp}{d\psi} \right)^2 \right] \frac{d\psi}{dp}
\]

\[
= p + \frac{1}{2} \cdot 2 \frac{dp}{d\psi} \cdot \frac{d^2 p}{d\psi^2} \frac{d\psi}{dp}
\]

\[
= p + \frac{d^2 p}{d\psi^2} \quad \left[ \therefore \frac{dp}{d\psi}, \frac{d\psi}{dp} = 1 \right]
\]

\[
\rho = p + \frac{d^2 p}{d\psi^2}
\]

is the required tangential polar formula.

**Solved Example 12.** Find the radius of curvature at any point on the curve \( p = a \sin \psi \cos \psi \).

**Solution.**

\[
p = a \sin \psi \cos \psi
\]

\[
\therefore \quad \frac{dp}{d\psi} = a \left( \cos^2 \psi - \sin^2 \psi \right)
\]

\[
\frac{d^2 p}{d\psi^2} = -2a \cos \psi \sin \psi - 2a \sin \psi \cos \psi
\]

\[
= -4a \sin \psi \cos \psi
\]

\[
\therefore \quad \rho = p + \frac{d^2 p}{d\psi^2} = a \sin \psi \cos \psi - 4a \sin \psi \cos \psi
\]

\[
= -3a \sin \psi \cos \psi
\]

\[
= -3p.
\]

**Solved Example 13.** Prove that for any curve

\[
\frac{r}{\rho} = \sin \phi \left( 1 + \frac{d\phi}{d\theta} \right)
\]
Solution. \( \psi = \theta + \phi, \quad r \frac{d\theta}{ds} = \sin \phi, \quad \frac{dr}{ds} = \cos \phi \)

Now
\[
\frac{1}{\rho} = \frac{d\psi}{ds} = \frac{d}{ds} = (\theta + \phi)
\]
\[
= \frac{d\theta}{ds} + \frac{d\phi}{ds}
\]
\[
\therefore \quad \frac{r}{\rho} = \frac{r}{ds} \frac{d\theta}{ds} + \frac{r}{ds} \frac{d\phi}{ds}
\]
\[
= \sin \phi + r \frac{d\phi}{d\theta} \frac{d\theta}{ds}
\]
\[
= \sin \phi + \frac{d\theta}{ds} \frac{d\phi}{d\theta}
\]
\[
= \sin \phi + \sin \phi \frac{d\phi}{d\theta}
\]
\[
= \sin \phi \left(1 + \frac{d\phi}{d\theta}\right)
\]

Hence the result.

Exercise–III
1. Find the radius of curvature at any point \((\psi, \theta)\) on the following curves:
   (i) \(r = \frac{a}{\theta}\)
   (ii) \(r = a \cos \theta\)
   (ii) \(r_1 = a^2 \cos \theta\)
   (iv) \(\frac{2a}{r}\)

2. Show that the radius of curvature at any point on the cardioid \(r = a (1 - \cos \theta)\) is \(\frac{2}{3} \sqrt{2ar}\).

3. Establish the formula
\[
\rho = \frac{\left(u^2 + u'^2\right)^{3/2}}{u'^2(u + u')} \quad \text{where} \quad u = \frac{1}{r}
\]
and dashes denote differentiation w.r.t. \(\theta\).

7. Newton’s Method

If a curve passes through the origin the radius of curvature at the origin is obtained by a special method which is due to Newton. For a curve passing through the origin and having X-axis as the tangent thereat, the radius of curvature at the origin is given by
\[ r_0 = \lim_{x \to 0} \left( \frac{x^2}{2y} \right) \]

Similarly, for a curve passing through the origin and having Y-axis as the tangent thereat the radius of curvature at the origin is given by the formulae

\[ r_0 = \lim_{y \to 0} \left( \frac{y^2}{2x} \right) \]

**Example.** Find the radius of curvature at the origin for the following curves

(i) \( x^2 = 2ay \)

(ii) \( y^2 = 4ax \)

**Solution.** For the curve \( x^2 = 2ay \), X-axis is the tangent at the origin

\[ r_0 = \lim_{x \to 0} \left( \frac{x^2}{2y} \right) = \lim_{x \to 0} \left( \frac{2ay}{2y} \right) = a \]

For the second curve \( y^2 = 4ax \), Y-axis is the tangent at (0, 0)

\[ r_0 = \lim_{x \to 0} \left( \frac{y^2}{2x} \right) = \lim_{x \to 0} \left( \frac{4ax}{2x} \right) = 2a \]

### 8. Centre of Curvature

The centre of curvature at any point on a curve is the limiting position of the point of intersection of the normal at P and the normal at a neighbouring point Q very close to P on the curve.

Let us find the coordinates of the centre of curvature at any point \((x, y)\) on the curve \( y = f(x) \).

Let \( Q(x + \Delta x, y + \Delta y) \) be a point on the curve close to P.

The equations of the normal at P and Q are respectively

\[
(Y - y) f'(x) + X - x = 0 \\
\text{[Y} - (y + \Delta y)] f'(x + \Delta x) + X - (x + \Delta x) = 0
\]

Subtracting (i) from (ii) \( X \) is eliminated and we get:

\[-(Y - y) f'(x) + [Y - y - \Delta y] f'(x + \Delta x) - \Delta x = 0\]

Or \( (Y - y) [f'(x + \Delta x) - f'(x)] = \Delta x + \Delta y f'(x + \Delta x) \)

Dividing throughout by \( \Delta x \) and proceeding to the limit as \( \Delta x \to 0 \), we get

\[ (Y - y) \frac{d}{dx} [f'(x)] = 1 + f(x) \cdot f'(x) \]

\[ \therefore \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x) \]

\[ i.e., \quad (Y - y) \ f''(x) = 1 + [f'(x)]^2 \]

\[ \therefore \quad Y = y + \frac{1 + [f'(x)]^2}{f''(x)} \]
\[ y + \frac{1}{d^2y}{dx^2} \] ... (iii)

Putting the value of \( Y \) in (\( i \)) or (\( ii \)) we get,

\[ X = x - (Y - y) \frac{dy}{dx} \]

\[ = x - \left[ 1 + \frac{d^2y}{dx^2} \right] \frac{dy}{dx} \]

\[ = x - \frac{d^2y}{dx^2} \] ... (iv)

The coordinates of the centre of curvature are given by (\( iii \)) and (\( iv \)).

9. **Circle of Curvature and Chord of Curvature**

The circle having the centre of curvature for its centre and the radius of curvature for its radius, is called the circle of curvature for the point under consideration. The centre and the radius being known, the equation of the circle of curvature can be written down.

The locus of centres of curvature of a curve is called the evolute and the given curve is called involute.

Any chord of the circle of curvature at a point on the curve passing through the point is called the chord of curvature.

**Solved Example 9.** Show that the centre of curvature at the origin lying on the parabola

\[ y = mx + \frac{x^2}{a} \] is given by

\[ \alpha = - \frac{1}{2} \frac{am}{a(1 + m^2)}, \beta = \frac{1}{2} a(1 + m^2) \]

Hence show that the equation of the circle of curvature is

\[ x^2 + y^2 = a (1 + m^2) (y - mx) \]

**Solution.**

\[ y = r \cdot x + \frac{x^2}{a} \] ... (i)

Here

\[ y_1 = m + \frac{2x}{a} \]

\[ \Rightarrow \]

\[ (y_1)_0 = m \]

\[ y_2 = \frac{2}{a} \]

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\( (y_2)_0 = \frac{2}{a} \)

\[ \rho = \frac{1 + n^2}{y_2} = \frac{(1 + m^2)^{3/2}}{2a} = \frac{1}{2}a(1 + m^2)^{3/2} \]

The coordinates of the centre of curvature at any point \((x, y)\) are given by the formulae

\[ \alpha = x - \frac{\frac{dy}{dx} \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]}{\frac{d^2y}{dx^2}} \]

\[ \beta = y + \frac{1 + \left( \frac{dy}{dx} \right)^2}{\frac{d^2y}{dx^2}} \]

At the origin,

\[ \alpha = 0 - \frac{m(1 + m^2)}{\frac{2}{a}} = -\frac{1}{2}am(1 + m^2) \]

\[ \beta = 0 + \frac{1 - m^2}{\frac{2}{a}} = \frac{1}{2}a(1 + m^2) \]

Equation of the circle of curvature is

\[(x - \alpha)^2 + (y - \beta)^2 = \rho^2 \]

i.e.,

\[ \left(x + \frac{1}{2}am(1 + m^2)\right)^2 + \left(y - \frac{1}{2}a(1 + m^2)\right)^2 = \frac{1}{4}a^2(1 + m^2)^3 \quad \therefore \rho = \frac{1}{2}a(1 + m^2)^{3/2} \]

Simplifying we get,

\[ x^2 + y^2 = a (1 + m^2) (y - mx) \]
1. Infinite Branches of a Curve

Curves like the parabola or the hyperbola are unlimited in extent and it is of great importance to know how they behave as one or more of their branches tend to infinity. It may be that an infinite branch might be approximated by straight line in the part of the $x$-$y$ plane very remote from the origin of coordinates. In our study of the curves, we are concerned with points and lines, etc. lying in the plane containing the coordinate axes of $x$ and $y$. We say that a point $P$ lies in the finite part of the plane, if the distance of $P$ from the origin is finite. Similarly a straight line, lying on the plane, is said to be in the finite part of the plane, if the perpendicular distance of the line from the origin is finite.

By a point, receding (or tending) to infinity on a curve, is meant a point moving on the curve such that its distance from the origin tends to infinity.

**Definition.** A straight line is said to be an **asymptote** of an infinite branch of a curve if it is at finite distance from the origin and if the perpendicular distance, from this line, of a point moving on the curve, tends to zero, as the point recedes to infinity along the infinite branch.

2. Asymptote to a General Curve

Let the equation of any curve be

$$\phi (x, y) = 0 \quad \ldots \quad (1)$$

In Figure 1 the asymptote $y = mx + c$ is approached when $x \to \infty$ along the curve. Then an asymptote of the curve, *not parallel* to the $y$-axis has an equation of the form

$$y = mx + c \quad \ldots \quad (2)$$

when $m$ and $c$ both are finite.
The perpendicular distance from the line (2) of a point \((x, y)\) on the branch of the curve to which the line (2), is an asymptote is given by the formula.

\[
p = \frac{y - mx - c}{\sqrt{1 + m^2}} \quad \ldots (3)
\]

where \(p\) must tend to zero when \(x \to +\infty\) or \(x \to -\infty\), or \(x \to +\infty\) and \(-\infty\) both.

Suppose that the line (2) is an asymptote, when \(x \to +\infty\). Since \(m\) is finite, therefore \(\sqrt{1 + m^2}\) is finite and non-zero (in fact greater than 1). Hence \(p\) tends to zero if

\[
\lim_{x \to +\infty} (y - mx - c) = 0 \quad \ldots (4)
\]

whence, on dividing by \(x\) we see that

\[
\lim_{x \to +\infty} \left( \frac{y - mx - c}{x} \right) = 0
\]

Since \(c\) is finite, \(\lim_{x \to +\infty} \frac{c}{x} = 0\) and hence, we get

\[
m = \lim_{x \to +\infty} \frac{y}{x} \quad \ldots (5)
\]

where the relation between \(x\) and \(y\) is \(\phi (x, y) = 0\), since the point \((x, y)\) lies on the given curve (1). Having obtained \(m\) from (5), the relation (4) then shows that

\[
c = \lim_{x \to +\infty} (y - mx)
\]

In Figure 2, the asymptote \(y = mx + c\) is approached when \(x \to -\infty\) along the curve.

For example, let the equation of a curve be \(y = e^{-x}\) and let us find its asymptote.
Let the equation of the asymptote be $y = mx + c$.

Here

$$m = \lim_{x \to -\infty} \frac{y}{x} = \lim_{x \to -\infty} \frac{e^{-x}}{x} = 0.$$  

[Note: that in this case $\lim_{x \to -\infty} \frac{y}{x} = \lim_{x \to -\infty} \left(\frac{e^{-x}}{x}\right)$ is not finite].

Also

$$c = \lim_{x \to -\infty} \left(y - mx\right) = \lim_{x \to -\infty} e^{-x} = 0.$$  

Hence the part of line $y = 0$ (where $x$ is positive) is an asymptote of the curve.

On the other hand, if the equation of the curve is $y = e^x$,

$$\lim_{x \to -\infty} \frac{y}{x} = \lim_{x \to -\infty} \frac{e^{-x}}{x} \text{ is not finite.}$$

But

$$m = \lim_{x \to -\infty} \frac{y}{x} = \lim_{x \to -\infty} \frac{e^{-x}}{x} = 0$$  

and

$$c = \lim_{x \to -\infty} \left(y - mx\right) = \lim_{x \to -\infty} e^x = 0.$$
Hence, the part of the line \( y = 0 \) (where \( x \) is negative) is asymptote to the curve \( y = e^x \). In this case, it is towards the negative side that the \( x \)-axis is an asymptote of the curve, where, as in the preceding case the \( x \)-axis is an asymptote towards the positive side.

Lastly, if the curve is

\[
y = mx + c + \frac{a + \sin x}{x}
\]

then

\[
m = \lim_{x \to +\infty} \frac{y}{x} = \lim_{x \to -\infty} \left( m + \frac{c}{x} + \frac{a + \sin x}{x^2} \right)
\]

as \( x \to +\infty \) or \( x \to -\infty \) in both the cases the limit is \( m \). Also \( y - mx \) tends to \( c \) as \( x \to +\infty \) or \( -\infty \).

Hence the asymptote to the curve (A) is the line \( y = mx + c \) whether \( x \to \infty \) or \( x \to -\infty \).

3. Asymptotes of an Algebraic Curve of the nth Degree IN \( X \) and \( Y \)

An algebraic curve of the \( n \)th degree in \( x \) and \( y \), can be always written in the form.

\[
\left[ a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \ldots + a_n y^n \right] + \left[ b_1 x^{n-1} + b_2 x^{n-2} y + b_3 x^{n-3} y^2 + \ldots + b_{n+1} y^{n-1} \right]
\]

\[
+ \left[ c_1 x^{n-2} + \ldots + c_{n-1} y^{n-2} \right] + \ldots + (Ax + By) + M = 0.
\]

in which the first bracket contains terms each of which is of the \( n \)th degree in \( x \) and \( y \) together \((i.e., \) homogeneous in \( x \) and \( y \) of degree \( n \)), the second bracket similarly contains terms each of which is of \( (n-1) \)th degree in \( x \) and \( y \) together, similarly other brackets contains terms of lower degree in \( x \) and \( y \) together.

Let us rewrite the term in the first bracket in the form

\[
x^n \left[ a_0 + a_1 \frac{y}{x} + a_2 \frac{y^2}{x^2} + \ldots + a_n \frac{y^n}{x^n} \right] \equiv x^n \phi_n \left( \frac{y}{x} \right)
\]

where \( \phi_n \left( \frac{y}{x} \right) \) stands for the algebraic sum of a number of terms in which the highest degree terms in \( \frac{y}{x} \) does not exceed \( n \). The equation (a) can therefore, be written as

\[
x^n \phi_n \left( \frac{y}{x} \right) + x^{n-1} \phi_{n-1} \left( \frac{y}{x} \right) + x^{n-2} \phi_{n-2} \left( \frac{y}{x} \right) + \ldots + x \phi_1 \left( \frac{y}{x} \right) + M = 0
\]

\[
\ldots (\alpha')
\]

**Note:** Had the terms in the first bracket consisted of only terms, viz.,

\[
a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2,
\]

the group of \( n \)th degree terms could have been expressed as

\[
x^n \left[ a_0 + a_1 \left( \frac{y}{x} \right) + a_2 \left( \frac{y}{x} \right)^2 \right]
\]

\[
\ldots (1)
\]
in which the highest degree of \( \frac{y}{x} \) within the box bracket [ ] is 2 (and not \( n \)). We notice in this case that if (1) is expressed as

\[ x^n \phi_n\left(\frac{y}{x}\right) \text{ then } \phi_n\left(\frac{y}{x}\right) \]

is a group of terms in which the highest degree of \( \frac{y}{x} \) is 2, which is supposed to be less than \( n \). We are expressing this group of three terms not as \( \phi_2\left(\frac{y}{x}\right) \) but as \( \phi_n\left(\frac{y}{x}\right) \), the reason being that \( \phi_n\left(\frac{y}{x}\right) \) is the co-efficient of \( x^n \). This is the justification for using the symbol \( x^n \phi_n\left(\frac{y}{x}\right) \) for the terms each of which is of the \( n \)th degree is \( x \) and \( y \). A similar interpretation is true for each term of (\( \alpha' \)).

Let us find out the asymptotes of (\( \alpha' \)) which are not parallel to the \( y \)-axis. We already know that such an asymptote is of the type \( y = mx + c \), where \( m \) and \( c \) are finite quantities, we also recall in our mind that

\[ m = \lim_{x \to \infty} \frac{y}{x} \text{ and } \lim_{x \to \infty} (y - mx) \]

These limits are to be found from the equation of the given curve itself.

From (\( \alpha' \)), by dividing both sides by \( x^n \), we get

\[ \phi_n\left(\frac{y}{x}\right) + \frac{1}{x} \phi_{n-1}\left(\frac{y}{x}\right) + \frac{1}{x^2} \phi_{n-2}\left(\frac{y}{x}\right) + \ldots + \frac{1}{x^{n-1}} \phi\left(\frac{y}{x}\right) + \frac{M}{x^n} = 0 \]

... (\( \alpha'' \))

Let \( x \to \infty \), then we have from (\( \alpha'' \))

\[ \phi_n\left(\frac{y}{x}\right) = 0 \]

i.e.,

\[ \phi_n\left(m\right) = 0 \]

... (\( \beta \))

[By writing \( \lim_{x \to \infty} \frac{y}{x} = m \)]

Let the roots of (\( \beta \)) be \( m_1, m_2, m_3, \ldots \). Consider the value \( m = m_1 \) and let the corresponding value of \( c \) denoted by \( c_1 \) then \( c_1 = \lim_{x \to \infty} (y - m_1 x) \)

To find the value \( c_1 \) from the given equation of the curve (\( \alpha' \)) let us proceed as follows:

Put \( y - m_1 x = p_1 \) where \( p_1 \) is a function of \( x \).

[Note: From the equation of the given curve (\( \alpha' \)) \( y \) is to be looked upon as an implicit function of \( x \) and thus \( y - m_1 x \) is a function of \( x, x \) and \( y \) being the co-ordinates of a point on (\( \alpha' \))]

When \( x \to \infty \), \( \lim p_1 = c_1 \).

From \( y - m_1 x = p_1 \) we find \( \frac{y}{x} = m_1 + \frac{p_1}{x} \). Substituting this value of \( \frac{y}{x} \) in the equation of the curve (\( \alpha' \)), we find
\[ x^n \phi_n \left( m_1 + \frac{p_1}{x} \right) + x^{n-1} \phi_{n-1} \left( m_1 + \frac{p_1}{x} \right) + x^{n-2} \phi_{n-2} \left( m_1 + \frac{p_1}{x} \right) + \cdots + x \phi \left( m_1 + \frac{p_1}{x} \right) + M = 0 \]

Expanding the functions

\[ \phi_n \left( m_1 + \frac{p_1}{x} \right), \phi_{n-1} \left( m_1 + \frac{p_1}{x} \right), \phi_{n-2} \left( m_1 + \frac{p_1}{x} \right), \cdots \]

with the help of Taylor’s theorems in ascending powers of \( \frac{p_1}{x} \), we get

\[ x^n \left[ \phi_n (m_1) + \frac{p_1}{x} \phi_n'(m_1) + \frac{1}{2!} \frac{p_1^2}{x^2} \phi_n''(m_1) + \cdots \right] + x^{n-1} \left[ \phi_{n-1} (m_1) + \frac{p_1}{x} \phi_{n-1}'(m_1) + \frac{1}{2!} \frac{p_1^2}{x^2} \phi_{n-1}''(m_1) + \cdots \right] + \cdots + x^1 \left[ \phi_1 (m_1) + \frac{p_1}{x} \phi_1'(m_1) + \frac{1}{2!} \frac{p_1^2}{x^2} \phi_1''(m_1) + \cdots \right] + M = 0 \]

Rearranging the terms in decending powers of \( x \), we get

\[ x^n \left[ \phi_n (m_1) + x \phi_n'(m_1) + \phi_{n-1}(m_1) \right] + x^{n-1} \left[ \frac{p_1}{x} \phi_n'(m_1) + \phi_{n-1}(m_1) \right] + x^{n-2} \left[ \frac{p_1^2}{2!} \phi_n''(m_1) + \frac{p_1}{x} \phi_{n-1}'(m_1) + \phi_{n-2}(m_1) \right] + \cdots \]

\[ + [\text{terms containing lower powers of } x] + M = 0 \]

\[ \text{Since } m_1 \text{ is a root of } \phi_n (m) = 0. \]

We get

\[ \phi_n (m_1) = 0 \]

\[ \therefore \quad (2) \text{ becomes} \]

\[ x^{n-1} \left[ p_1 \phi_n'(m_1) + \phi_{n-1}(m_1) \right] + x^{n-2} \left[ \frac{p_1^2}{2!} \phi_n''(m_1) + \frac{p_1}{x} \phi_{n-1}'(m_1) + \phi_{n-2}(m_1) \right] + \cdots \]

\[ + \cdots + \cdots \cdots + M = 0 \]

Dividing both sides of (3) by \( x^{n-1} \), we get

\[ \left[ p_1 \phi_n'(m_1) + \phi_{n-1}(m_1) \right] + \frac{1}{x} \left[ \frac{p_1^2}{2!} \phi_n''(m_1) + \frac{p_1}{x} \phi_{n-1}'(m_1) + \phi_{n-2}(m_1) \right] + \cdots \]

\[ + \cdots + \frac{M}{x^{n-1}} = 0 \]
Let $x \to \infty$. Since according to our assumption
\[
\lim_{x \to \infty} p_1 = c_1, \text{ we get from (4)}
\]
\[
c_1 \phi_n'(m_1) + \phi_{n-1}(m_1) = 0
\] ... (5)

\[
\therefore \quad c_1 = \frac{\phi_{n-1}(m_1)}{\phi_n'(m_1)}, \text{ provided } \phi_n'(m_1) \neq 0.
\]

Hence when $m = m_1$ we get the asymptote
\[
y = m_1 x - \frac{\phi_{n-1}(m_1)}{\phi_n'(m_1)}, \text{ if } \phi_n'(m_1) \neq 0.
\]

Similarly when $m = m_2$, we get as before an asymptote
\[
y = m_2 x - \frac{\phi_{n-1}(m_1)}{\phi_n'(m_1)}, \text{ if } \phi_n'(m_1) \neq 0.
\]

It is supposed here that the roots of (B) [viz. $\phi_n(m) = 0$] are all real and unequal i.e., no two of them are equal.

**Case I**

If in (5) $\phi_n'(m_1) = 0$, but $\phi_{n-1}(m_1) \neq 0$

no value of $c_1$ can be obtained. In this case there is no asymptote of the curve for the value $m = m_1$.

If, however, both $\phi_n'(m_1) = 0$ and $\phi_{n-1}(m_1) = 0$, the equation (5) becomes an identity. To determine the value of $c_1$ in this, let us proceed from (4) which is, therefore, written on multiplying by $x$ as

\[
\left[ \frac{p_1^2}{2!} \phi_n''(m_1) + p_1 \phi_n'(m_1) + \phi_{n-2}(m_1) \right] + \frac{1}{x} \left[ \frac{p_1^3}{3!} \phi_n'''(m_1) + \frac{p_1^2}{2!} \phi_n''(m_1) + p_1 \phi_n'(m_1) + \phi_{n-3}(m_1) \right] + \ldots + \frac{M}{x^{n-2}} = 0 \quad \cdots (4')
\]

Now, let $x \to \infty$. Then since $\lim_{x \to \infty} p_1 = c_1$, we get from (4').

\[
\frac{c_1^2}{2!} \phi_n''(m_1) + c_1 \phi_n'(m_1) + \phi_{n-2}(m_1) = 0 \quad \cdots (A)
\]

which is a quadratic in $c_1$ to determine two values of $c_1$ [provided $\phi_n''(m_1) \neq 0$].

Let the roots of (A) be real unequal viz., $c_1'$ and $c_1''$. In this case we get two parallel asymptotes.

\[
y = m_1 x + c_1'.
\]

\[
y = m_1 x + c_1''.
\]
Note: We get two parallel asymptotes for the value \( m = m_1 \) only when \( \phi''_n(m_1) \neq 0 \).

\[ \phi'_n(m_1) = 0 \text{ and also } \phi_{n-1}(m_1) = 0 \]

But if also \( \phi''_n(m_1) = 0 \) then the value of \( c_1 \) is obtained as above from a cubic equation in \( c_1 \) viz.,

\[ \frac{c_1^3}{3!} \phi''_n(m_1) + \frac{c_1^2}{2!} \phi'_n(m_1) + c_1 \phi'_{n-2}(m_1) + \phi_{n-3}(m_1) = 0 \]

... (B)

provided \( \phi''_n(m_1) \neq 0 \).

In this case, if the three roots are real and distinct we have three parallel asymptotes for the value \( m = m_1 \) viz.

\[ y = m_1x + c'_1 \]
\[ y = m_1x + c''_1 \]
\[ y = m_1x + c'''_1 \]

where \( c'_1, c''_1 \) and \( c'''_1 \) are the roots of (B).

Deductions made from the Equation B viz., \( \phi_n(m) = 0 \)

(1) We notice that expression \( \phi_n(m) \) is obtained from the \( n \)th degree terms of the algebraic curve (\( \alpha \)) viz., the terms of \( x^n \phi_n \left( \frac{y}{x} \right) \) by putting \( x = 1 \) and \( y = m \).

Since, as already pointed out the degree of \( \phi_n \left( \frac{y}{x} \right) \) in \( \frac{y}{x} \) is never greater than \( n \) (i.e., is either less than \( n \) or at the most equal to \( n \). The degree of \( \phi_n(m) \) is at the most \( n \). Hence the equation \( \phi_n(m) = 0 \) has at the most \( n \) roots in \( m \). Since the equation \( \phi_n(m) = 0 \) determines the direction of asymptotes not parallel to the \( y \)-axis, we conclude that in algebraic curve of the \( n \)th degree given by (\( \alpha \)) can have at the most \( n \) asymptotes not parallel to the \( y \)-axis.

(2) Since the determination of \( m \) and \( c \) (for asymptotes not parallel to the \( y \)-axis) depends in general upon the equations.

\( i \) \( \phi_n(m) = 0 \)

\( ii \) \( c\phi'_n(m) + \phi_{n-1}(m) = 0 \)

and since \( \phi_n(m) \) [and hence \( \phi'_n(m) \) and \( \phi_{n-1}(m) \) ] depend upon \( x^n \phi_n \left( \frac{y}{x} \right) \) and \( x^{n-1} \phi_{n-1} \left( \frac{y}{x} \right) \).

We conclude that in general the determination of asymptotes not parallel to the \( y \)-axis, of an algebraic curve of the \( n \)th degree depends upon the \( n \)th degree and \((n-1)\)th degree terms of the algebraic curve.

Solved Example 1. Find the asymptotes of the curve

\[ y^3 - x^2y - 2xy^2 + 2x^3 - 7xy + 3y^2 + 2x^2 + 2x + 2y + 1 = 0 \]

Solution. Putting \( x = 1, \ y = m \) in the third and second degree terms separately, we get

\[ \phi_3(m) = m_3 - 2m^2 - m + 2 \]
\[
\begin{align*}
&= (m - 1) (m^2 - m - 2) \\
&= (m - 1) (m + 1) (m - 2), \\
\phi_2 (m) &= -7m + 3m^2 + 2 \\
\phi_3'(m) &= 3m^2 - 4m - 1
\end{align*}
\]
Now \[
\phi_3 (m) = m^3 - 2m^2 - m + 2 = 0 \text{ gives } m = 1, -1 \text{ or } 2.
\]
But \[
c = \left. \frac{\phi_1 (m)}{\phi_3'(m)} \right| = \frac{-7m + 3m^2 + 2}{3m^2 - 4m - 1}
\]
\[
\therefore \text{ For } m = 1, \quad c_1 = \frac{\phi_1 (1)}{\phi_3'(1)} = \frac{-2}{(-2)} = -1.
\]
For \[
m = -1, \quad c_2 = \frac{\phi_1 (-1)}{\phi_3'(-1)} = \frac{12}{6} = 2.
\]
For \[
m = 2, \quad c_3 = \frac{\phi_1 (2)}{\phi_3'(2)} = \frac{0}{3} = 0.
\]
The three asymptotes are, therefore

(i) \( y = x - 1 \), (ii) \( y = -x - 2 \), and (iii) \( y = 2x \).

**Solved Example 2.** Find the asymptotes of the curve

\[ x^3 + 3x^2y - 4y^3 - x + y + 3 = 0. \]

**Solution.** The given equation is

\[
x^3 \left[ 1 + 3 \frac{y}{x} - 4 \left( \frac{y}{x} \right)^3 \right] + x \left( \frac{y}{x} - 1 \right) + 3 = 0
\]

i.e., of the form \( x^3 \phi_1 \left( \frac{y}{x} \right) + x \phi_2 \left( \frac{y}{x} \right) + 3 = 0 \)

Thus \( \phi_3 (m) = -4m^3 + 3m + 1, \quad \phi_2 (m) = 0. \)

and \( \phi_1 (m) = m - 1. \)

The gradients \( m \) of the asymptotes are given by

\[
\phi_3 (m) = 0
\]

or \( 4m^3 - 3m - 1 = 0 \)

or \( (m - 1) (4m^2 + 4m + 1) = 0 \)

or \( (m - 1) (2m + 1)^2 = 0. \)

Thus there are three asymptotes, one having a gradient \( m = 1 \) and the other two have the same gradient \( m = -\frac{1}{2} \) and therefore are parallel.
To find $c$, we have the equation
\[ c\phi'_1(m) + \phi_2(m) = 0 \] ... (1)

But $\phi'_1(m) = -12m^2 + 3$ and $\phi_2(m) = 0$

when
\[ m = 1, \quad \phi'_1(1) = -9 \]

and when
\[ m = -\frac{1}{2}, \quad c\phi'_1\left(-\frac{1}{2}\right) = -3 + 3 = 0. \]

Thus when $m = 1$ then
\[ c = -\frac{\phi_2(1)}{\phi'_1(1)} = 0 \text{ and the asymptote is } y = x. \]

When $m = -\frac{1}{2}$, then the equation determining $c$ viz., the equation (1) becomes an identity.

We do not get any value of $c$ from this equation.

To determine the value of $c$, we have to consider the equation
\[ \frac{c^2}{2!}\phi'_1(m) + c\phi'_2(m) + \phi_1(m) = 0 \]
i.e.,
\[ \frac{1}{2}c^2(-24m)c.0 + (m-1) = 0, \]

Thus when $m = -\frac{1}{2}$, we get
\[ 6c^2 - \frac{3}{2} = 0 \]
or
\[ c^2 = \frac{1}{4} \]
\[ \therefore \quad c = \pm \frac{1}{2}. \]

Hence the other two asymptotes are
\[ y = -\frac{x}{2} \pm \frac{1}{2} \]
or
\[ 2y + x = \pm 1. \]

Exercise–I
1. Find the asymptotes of the following curves:
   (i) $y^3 + 3y^2x - x^3y - 3x^3 + y^2 - 2xy + 3x^2 + 4y + 5 = 0$
   (ii) $y^3 + x^2y + 2xy^2 - y + 1 = 0$
(iii) \( y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 = 0 \)

(iv) \( (x + y)(x + 2y + 2) = x + 9y + 2 \)

(v) \( 2x^3 - x^2y + 2xy^2 + y^3 - 4x^2 + 8xy - 4x + 1 = 0 \)

4. Determinations of Asymptotes Parallel to the \( Y \)-axis of an Algebraic Curve

If there is a curve having an asymptote parallel to the \( y \)-axis as shown in the diagram, the perpendicular distance from a point on the curve upon the line \( x = c \) which is parallel to the \( y \)-axis diminishes to zero as \( y \to +\infty \) or \( y \to -\infty \) and \(-\infty\) both.

Let the equation of the curve \((a)\) in art 3 of this lesson be written in the following form

\[
a_n y^n + y^{n-1}(a_{n-1} + b_{n-1}) + y^{n-2}(a_{n-2}x^2 + b_{n-2}x + c_{n-2}) + \ldots = 0 \quad \ldots (d)
\]

In Fig. 5 the asymptote \( x = c \) is approached when \( y \to +\infty \) along the curve I

In Fig. 6 the asymptote \( x = c \) is approached when \( y \to -\infty \) along the curve II

Case I

Let us suppose now that \( a_n \neq 0 \) \( i.e., \) there is no term in \((a)\) containing the term \( y^n \). Then, dividing every term on both sides of \((d)\) by \( y^{n-1} \), we get

\[
\left( a_{n-1}x + b_{n-1} \right) + \frac{1}{y} \left( a_{n-2}x^2 + b_{n-2}x + c_{n-2} \right) + \frac{1}{y^2} \left( a_{n-3}x^3 + b_{n-3}x^2 + c_{n-3}x + d_{n-3} \right) + \ldots = 0 \quad \ldots (d')
\]

Let \( y \to \infty \) then assuming that \( a_{n-1} \neq 0 \) the equation \((d')\) reduces to

\[
a_{n-1}x + b_{n-1} = 0.
\]

\( i.e., \)

\[
x = -\frac{b_{n-1}}{a_{n-1}}.
\]
Thus we get the asymptote \( x = -\frac{b_{n-1}}{a_{n-1}} \) which is a line parallel to the \( y \)-axis.

**Case II**

In case not only \( a_n = 0 \), but \( a_{n-1} = 0 \) and also \( b_{n-1} = 0 \) the equation (\( \delta \)) after dividing both sides by \( y^{n-2} \), becomes

\[
\left(a_{n-2}x^2 + b_{n-2}x + c_{n-2}\right) + \frac{1}{y}\left(a_{n-3}x^3 + b_{n-3}x^2 + c_{n-3}x + d_{n-3}\right) + \ldots = 0 \quad \ldots (\delta'')
\]

Let \( y \to \infty \) then assuming that \( a_{n-3} \neq 0 \), we find that (\( \delta''' \)) is reduced to

\[
a_{n-2}x^2 + b_{n-2}x + c_{n-2} = 0 \quad \ldots (\delta''')
\]

or we get the asymptotes given by (\( \delta''' \)).

If the roots of this equation are real and different, we get two parallel asymptotes (each parallel to the \( y \)-axis) like \( x = \alpha_1 \) and \( x = \alpha_2 \), where \( \alpha_1 \) and \( \alpha_2 \) are two roots of (\( \delta''' \)). A generalization of this case, where the curve may have three or more than three asymptotes parallel to the \( y \)-axis can be easily made by the student. We therefore conclude that if the equation of Art 3 does not contain the term involved \( y^n \) but the coefficient of \( y^{n-1} \) is a linear function of \( x \) viz., \( a_{n-1}y + b_{n-1} \) (where \( a_{n-1} \neq 0 \)) then there is one asymptote parallel to the \( y \)-axis, viz.,

\[
y = -\frac{b_{n-1}}{a_{n-1}}.
\]

But if \( a_n = 0, a_{n-1} = 0, b_{n-1} = 0 \) and the coefficient of \( y^{n-2} \) is a quadratic expression in \( x \), (viz., \( a_{n-2}x^2 + b_{n-2}x + c_{n-2} \)), where \( a_{n-2} \neq 0 \) then there are two asymptotes parallel to the \( y \)-axis whose joint equation given by \( a_{n-2}x^2 + b_{n-2}x + c_{n-2} = 0 \) (provided it has real and different roots).

An exactly similar method may be adopted to determine the asymptotes of a curve parallel to the \( x \)-axis (if there be any). Thus if in (\( \alpha \)) in Art 3 of this, lesson \( a_0 = 0 \) (i.e., If there is no term involving \( x^n \)) and if the coefficient of \( x^{n-1} \) viz., \( a_1y + b_1 \) is equated to zero (provided \( a_2 \neq 0 \)), we get the asymptote

\[
y = -\frac{b_1}{a_1} \quad \text{which is parallel to the } x \text{-axis.}
\]

In case \( a_0 = 0, a_1 = 0 \) and \( b_1 = 0 \) in Art 3, the coefficient of \( x^{n-2} \) viz., \( a_2y^2 + b_2y + c_2 \) equated to zero is the joint equation of the two asymptotes parallel to the \( x \)-axis.

A generalization of this case where the curve may have three or more than three asymptotes parallel to the \( x \)-axis can be made by student.

**Solved Example 3.** Write down the equations of the asymptotes parallel to the co-ordinate axes of \( x^2y^3 + x^3y^2 + x^4 + y^3 \).

**Solution.** It is a curve (algebraic curve) of the 5th degree in \( x \) and \( y \).

We rewrite the equation of the curve in descending powers of \( y \) as

\[
0.y^5 + 0.y^4 + y^3 (x^2 - 1) + y^2x^3 - x^4 = 0.
\]

The highest power of \( y \) present in the equation is 3. Equating to zero the coefficient of \( y^3 \) which is a quadratic expression in \( x \), in this case, we get

\[
x^2 - 1 = 0, \ i.e., \ x = \pm 1
\]

which are the asymptotes parallel to the \( y \)-axis.
The highest power of $x$ present in the equation is 4 and the co-efficient of $x^4$ when the equation is arranged in powers of $x$ is a mere constant (viz., 1), so that there is no asymptote parallel to the $x$-axis.

**Solved Example 4.** Find the asymptotes, parallel to the $x$-axis of the curve

$$y^4 + x^2y^2 + 2xy^3 - 4x^2 - y + 1 = 0$$

**Solution.** It is a curve of the fourth degree.

There are no terms containing $x^4$ and $x^3$ and the co-efficient of $x^2$ is a quadratic function of $y$ (viz., $y^2 - 4$).

Equating $y^2 - 4 = 0$, we get the asymptotes $y = \pm 2$, which are parallel to the $x$-axis.

5. **Asymptotes in Polar Coordinates**

If $\theta_1$ be a root of the equation $f(\theta) = 0$, [where $f(\theta)$ is a differentiable function of $\theta$], then the asymptote of the curve $\frac{1}{r} = f(\theta)$, is the line

$$\frac{1}{r} = f'(\theta_1) \sin (\theta - \theta_1) \quad \ldots \quad (1)$$

Let

$$p = r \cos (\theta - \alpha)$$

be the equation of an asymptote to the curve

$$\frac{1}{r} = f(\theta) \quad \ldots \quad (2)$$

on which P is any point $(\phi, \theta)$.

Then as P recedes to infinity on the curve, i.e., as $r \to \infty$, $PM$, the perpendicular from P on the asymptote (1), tends to zero.
Draw OL (= p) the perpendicular from the pole O on the asymptote such that \( \angle \text{LOX} = \alpha \) and from P draw PK perpendicular on OL. Then

\[
\text{PM} = \text{KL} = \text{OL} - \text{OK} = p - \text{OP} \cos \angle \text{POL} = p - r \cos (\theta - \alpha)
\]

Since PM tends to zero as P recedes to infinity on the curve,

\[
\therefore \quad \lim_{r \to \infty} \text{PM} = 0
\]

i.e.,

\[
\lim_{r \to \infty} \left[ p - r \cos(\theta - \alpha) \right] = 0 \quad \ldots \; (3)
\]

or

\[
\lim_{r \to \infty} \cos(\theta - \alpha) = \lim_{r \to \infty} \left( \frac{p}{r} \right) = 0
\]

since \( p \) is finite.

Here

\[
\lim_{r \to \infty} (\theta - \alpha) = \frac{\pi}{2}
\]

i.e.,

\[
\lim_{r \to \infty} \theta = \alpha + \frac{\pi}{2}
\]

i.e.,

\[
\alpha = \lim_{f(\theta) = 0} \theta - \frac{\pi}{2}
\]

Hence if \( f(\theta_1) = 0 \), i.e., if \( \theta_1 \) is a root of \( f(\theta) = 0 \), then

\[
\alpha = \theta_1 - \frac{\pi}{2} \quad \ldots \; (4)
\]

From (3),

\[
p = \lim_{r \to \infty} r \cos(\theta - \alpha)
\]

\[
= \lim_{\theta \to \theta_1} \frac{\cos \left[ \theta - \left( \theta_1 - \frac{\pi}{2} \right) \right]}{f(\theta)}, \quad \text{by (2) and (4)}
\]

\[
= \lim_{\theta \to \theta_1} \frac{-\sin(\theta - \theta_1)}{f(\theta)} = \lim_{\theta \to \theta_1} \frac{\sin(\theta_1 - \theta)}{f(\theta)}
\]

... (5)

In the last fraction of (5) both the numerator and denominator tend to zero when \( \theta \to \theta_1 \). To evaluate this limit we differentiate both the numerator and denominator and then put \( \theta = \theta_1 \). Thus

\[
p = \lim_{\theta \to \theta_1} \left[ \frac{d}{d\theta} \frac{\sin(\theta_1 - \theta)}{f(\theta)} \right]
\]

\[
= \lim_{\theta \to \theta_1} \frac{-\cos(\theta_1 - \theta)}{f'(\theta)} = \frac{1}{f'(\theta_1)}
\]
\[ \lim_{x \to a} \frac{f'(x)}{F'(x)} = \frac{f'(a)}{F'(a)}, \text{ if } F'(a) \neq 0 \text{ and } \lim_{x \to a} f(x) = 0 = \lim_{x \to a} F(x) \]

Hence the equation (1) of the asymptote becomes

\[ -\frac{1}{f'(\theta_i)} = r \cos \left( \theta - \theta_i - \frac{\pi}{2} \right) \]

or

\[ -\frac{1}{f'(\theta_i)} = r \sin (\theta_i - \theta) \]

\[ i.e., \quad \frac{1}{r} = f'(\theta_i) \sin (\theta - \theta_i) \]

Solved Example 5. Find the asymptote of the curve.

\[ r\theta = a. \]

Solution. Writing the equation in the form \( \frac{1}{r} = f(\theta) \), we see that

\[ f(\theta) = \frac{1}{r} = \frac{\theta}{a} \]

The root of the equation \( f(\theta) = 0 \) is given by \( \theta = 0 \).

Now

\[ f'(\theta) = \frac{1}{a} = \text{so that } f'(0) = \frac{1}{a} \]

The equation of the asymptote is, therefore

\[ \frac{1}{r} = f'(0) \sin (\theta - 0) \]

or

\[ \frac{1}{r} = \frac{1}{a} \sin \theta \]

or

\[ r \sin \theta = a. \]

Solved Example 6. Find the asymptotes of the curve

\[ r \cos \varnothing = a \sin 3\theta. \]

Solution. Writing the equation in the form \( \frac{1}{r} = f(\theta) \) we see that

\[ f(\theta) = \frac{1}{r} = \frac{\cos 2\theta}{a \sin 3\theta} = \frac{1}{a} \cos 2\theta \csc 3\theta \]

The root of the equation \( f(\theta) = 0 \) are given by

\[ \cos \varnothing = 0 = \cos \frac{\pi}{2}. \]
so that \[ 2\theta = n\pi + \frac{\pi}{2}, \text{ i.e., } \theta = \frac{n\pi}{2} + \frac{\pi}{4}. \]

Consider \[ \theta = \frac{\pi}{4}, \text{ and } \frac{\pi}{4}. \]

Also \[ f'(\theta) = \frac{1}{a}[-2\sin 2\theta \csc 3\theta - 3\cos 2\theta \csc 3\theta \cot 3\theta] \]

\[ = -\frac{1}{a}[2\sin 2\theta + 3\cos 2\theta \cot 3\theta] \csc 3\theta \]

so that \[ f'(\frac{\pi}{4}) = -\frac{2}{a}\sqrt{2} = f'(\frac{-\pi}{4}). \]

The asymptotes are, therefore,

\[ \frac{1}{r} = -\frac{2\sqrt{2}}{a} \sin \left( \theta - \frac{\pi}{4} \right) = \frac{-2}{a} (\sin \theta - \cos \theta) \]

and

\[ \frac{1}{r} = -\frac{2\sqrt{2}}{a} \sin \left( \theta + \frac{\pi}{4} \right) = \frac{-2}{a} (\sin \theta + \cos \theta) \]

\[ 2r \ (\cos \theta - \sin \theta) = a \text{ and } a + 2r \ (\cos \theta + \sin \theta) = 0. \]

**Exercise–II**

Find the asymptotes of the following curves:

1. \[ y^2 (x^2 - a^2) = x \]
2. \[ x^3 y^2 = a^2 \ (x^2 + y^2) \]
3. \[ r \sin \theta = 2 \cos \theta \]
4. \[ r = a \cosec \theta + b \]
5. \[ r \sin n\theta = a. \]
Introduction

We know that ordinarily at a point on a curve, there is only one tangent and the arc of the curve in a small neighbourhood of the point is wholly on one side of the tangent. Also in general the derivative at a point \( P \) (i.e., the slope of the tangent at \( P \)) has a unique value. A point not possessing these properties is called a singular point and it is our purpose in this lesson to investigate such point on a curve.

**Definitions.** A point on a curve through which more than one branch of the curve passes is called a multiple point. Multiple points are examples of singular points on a curve. A point on a curve through which two branches of curve passes is called a double point. If more than two branches of a curve pass through a point we have multiple points of higher orders.

If \( r \) branches of a curve pass through a point we have a multiple point of the \( r \)th order. In this lesson, however, we shall confine our attention to double points only.

Classification of Double Points

A curve must have two tangents at a double point, one to each of the branches which pass through the double point. There are three types of double points viz., Node, Cusp and Conjugate point.

A double point on a curve is called a Node if two tangents at the double point are real and distinct.

A double point on a curve is called a Cusp if the two tangents at the double point are coincident.

A double point on a curve is called a conjugate point, if the point lies on the curve, but there is no point in the immediate neighbourhood which lies on the curve. In other words there are no real tangents to the two branches of the curve passing through the point.

To investigate the nature of a double point on a curve, we have to examine the nature of tangents at the point to the two branches of the curve. Ordinarily if we take any point \( P \) on a curve, the curve extends on either side of the point and the two arcs in Fig. 1 do not constitute two branches one would find that the value of the derivative at \( P \) obtained as a limit when \( Q \rightarrow P \) is the same as that obtained when \( Q' \rightarrow P \). When this is not so, the point is a multiple point.

In the case of a node the very first derivative

\[
\frac{dy}{dx} \quad \text{(or} \quad \frac{ds}{d\phi} \quad \text{or any other)}
\]

has different values when the moving point tends to the node along different branches passing through it.

In the case of a Cusp, the second derivatives at the Cusp on the two branches of the curve are different.
Tangents at the Origin

In order to investigate the nature of double points the first thing is to find the tangents there. For this the following proposition is very useful.

If a curve given by a polynomial in \( x \) and \( y \) passes through the origin, then the equation of the tangents at the origin is obtained by equating to zero the lowest degree terms in the polynomial.

Let the equation of any algebraic curve passing through the origin be

\[
(ax + by) + (a_2x^2 + b_2xy + c_2y^2) + (a_3x^3 + b_3x^2y + c_3xy^2 + d_3y^3) + \ldots = 0 \quad \ldots (1)
\]

where in (1) the constant term is absent, as the curve passes through the origin.

If \( P(x, y) \) is any point very close to \( O \) on the curve, the gradient of \( OP \) is \( \frac{y}{x} \). Now \( P \to O \) the limiting position of \( OP \) is the tangent at \( O \).

\[ \therefore \text{the slope of the tangent at } O \text{ is given by} \]

\[ m = \lim_{x \to 0, y \to 0} \left( \frac{y}{x} \right) \]

Dividing (1) by \( x \), we have

\[ a_1 + b_1 \left( \frac{y}{x} \right) + x \left[ a_2 + b_2 \left( \frac{y}{x} \right) + c_2 \left( \frac{y}{x} \right)^2 \right] + x^2 \left[ a_3 + b_3 \left( \frac{y}{x} \right) + c_3 \left( \frac{y}{x} \right)^2 + d_3 \left( \frac{y}{x} \right)^3 \right] + \ldots = 0 \]

Now taking limits as \( x \to 0 \) we get,

\[ a_1 + b_1 \lim_{x \to 0, y \to 0} \left( \frac{y}{x} \right) = 0 \quad \Rightarrow \quad \lim_{x \to 0, y \to 0} \left( \frac{y}{x} \right) = -\frac{a_1}{b_1} \]

where \( a_1 \neq 0 \).
Equation of tangent at O is

\[ y = \frac{a_x}{b_y} x \text{ or } a_x x + b_y y = 0 \]

which is the same as obtained from (1) by equating to zero the lowest degree terms.

In case \( a_1 = 0, b_1 = 0 \), when equation (1) becomes

\[
(a_x x^2 + b_x xy + c_x y^2) + (a_y x^3 + b_y x^2 y + c_y xy^2 + d_y y^3)
\]

........................ + ........................................ = 0  \hspace{1cm} \text{... (2)}

Dividing by \( x^2 \) and taking limits as \( x \to 0 \) we get

\[ a_2 + b_2 \lim_{y \to 0} \left( \frac{y}{x} \right) + c_2 \lim_{y \to 0} \left( \frac{y^2}{x} \right) = 0 \]

\[ \therefore \text{ The slopes of the tangent at O are given by the quadratic.} \]

\[ a_2 + b_2 m + c_2 m^2 = 0 \]

This is a quadratic to \( m \) giving two values of \( m \). Hence there are two tangents at the origin. The joint equation of the tangents is given by

\[ a_2 x^2 + b_2 xy + c_2 y^2 = 0 \]

which could have been obtained by equating to zero the lowest degree terms in (2).

If \( a_2, b_2, c_2 \) are all zero, then there are no second degree terms in the equation of the curve. Proceeding as above we can still show that the equation of the tangents at the origin can be obtained by equating to zero the lowest degree terms in the equation of the curve. Hence the result.

**Cor.** The origin is a multiple point on a rational algebraic curve only if the equation of the curve does not contain the constant and the first degree terms.

**Nature of the Origin Supposed to be a Point on an Algebraic Curves**

According to the nature of tangents at the origin, algebraic curves may by classified as follows:

(i) If at least one first degree term is present in the equation of the curve there is one tangent at the origin and the origin is an ordinary point on the curve.

(ii) If the terms of the lowest degree in the equation are quadratic terms, then origin is a double point. There is a conjugate point at the origin if the quadratic terms have no real factor. If the terms, have distinct factors, then origin is a node and if the terms have coincident factors \( i.e., \) they form a perfect square, then origin is a cusp.

**Solved Example, 1.** Examine the nature of the origin on the curve

\[ y^2 = 2x^2 y + x^3 y - 2x^2 \]

**Solution.** Transposing the terms on the right to the left, the expression formed by the lowest degree terms is found to be \( y^2 + 2x^2 \), which is a quadratic having no real factors. Hence the origin is a conjugate point on the given curve.

To examine the nature of singularity of any other point on an algebraic curve, transfer the origin to the given point and examine the new origin.

**Solved Example 2.** Prove that the curve

\[ ay^2 = (x - a) (x - b) \]
has at \( x = a \), a conjugate point if \( a < b \), a node if \( a > b \) and a cusp if \( a = b \).

**Solution.** Shifting the origin to the point \((a, 0)\) the equation of the curve becomes
\[ ay^2 = x^2 (x + a - b) \]

The expression formed by the lowest degree terms is \( ay^2 - (a - b)x^2 \). This quadratic has imaginary factors if \( a < b \) and then the point is a conjugate point. It has real and distinct factors if \( a > b \) and then the point is a node. It is a perfect square if \( a = b \) in which case the point is a cusp.

**Search for Double points**

If instead of the nature of a given point on an algebraic curve being required to be investigated, it is desired to search for double points on any curve whether algebraic or transcendental, the following method is used.

Let the equation of any curve be
\[ f(X, Y) = 0 \]
and suppose that \( P(x, y) \) is a double point on it. Then since \((x, y)\) satisfies the equation of the curve we have
\[ f(x, y) = 0 \quad \text{... (1)} \]

The tangent at \((x, y)\) is
\[ Y - y = \frac{dy}{dx} (X - x) \]

where \( \frac{dy}{dx} \) at \((x, y)\) is given by the relation.
\[ \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} = 0 \quad \text{... (2)} \]

If at least one of the two viz., \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) is non-zero, there is a single tangent at the point \( P \) and the point \( P \) is therefore, an ordinary point.

If both, \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) are zero at \( P \), the equation (2) is satisfied by more than one value of \( \frac{dy}{dx} \).

Thus we see that a necessary and sufficient condition for any point \( P(x, y) \) on a curve \( f(X, Y) = 0 \) to be a multiple point are that \( \frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0 \).

In such a case we differentiate the relation (2) to find \( \frac{dy}{dx} \), getting,
\[ \frac{\partial^2 f}{\partial^2 x^2} + \frac{\partial^2 f}{\partial y \partial x} \frac{dy}{dx} + \frac{\partial f}{\partial y} \frac{d^2 y}{dx^2} + \left( \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} \right) \frac{dy}{dx} = 0 \]

Since \( \frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0 \)
The above equation becomes a quadratic in \( \frac{dy}{dx} \).

\[
\frac{\partial^2 f}{\partial x^2} + 2 \frac{\partial^2 f}{\partial y \partial x} \frac{dy}{dx} + \frac{\partial^2 f}{\partial y^2} \left( \frac{dy}{dx} \right)^2 = 0
\]

\( \text{viz.} \) \[ \frac{d^2 y}{dx^2} = 0 \] \( \ldots (3) \)

assuming that \( \frac{d^2 y}{dx^2} \) at \( P \) is not infinite.

The roots of (3) are real and distinct and consequently there is a node at \( P \), if the discriminant of (3) is positive \( i.e., \) if

\[
\left( \frac{\partial^2 f}{\partial y \partial x} \right)^2 > \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2}
\]

The roots of (3) are equal and consequently there is a cusp at \( P \), if the discriminant of (3) is zero \( i.e., \) if

\[
\left( \frac{\partial^2 f}{\partial y \partial x} \right)^2 = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2}
\]

The roots of (3) are imaginary and consequently the point \( P \) is a conjugate point if the discriminant is negative, \( i.e., \) if

\[
\left( \frac{\partial^2 f}{\partial y \partial x} \right)^2 < \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2}
\]

**Working Rule:** To search for double points on a given curve \( f(x, y) = 0 \).

**Step I.** Solve the equations \( \frac{df}{dx} = 0, \frac{df}{dy} = 0, \)

Partially w.r.t.x, y the given equation \( f(x, y) = 0 \) and retain those solutions which satisfy the given equation \( f(x, y) = 0 \).

**Step II.** Find \( \left[ \frac{\partial^2 f}{\partial x \partial y} \right]^2 - \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} \) at the above points.

**Step III.** The point \( (x, y) \) is a node, cusp or conjugate point according as this quantity is positive, zero or negative.

The case when all the partial derivatives \( \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y \partial x} \) and \( \frac{\partial^2 f}{\partial y^2} \) are zero, indicates that \( (x, y) \) is a multiple point of 3rd or higher order. Investigation of such points is not in present course of study.

**Solved Example 3.** Determine the existence and nature of double points on the curve

\( (x - 2)^2 = y(y - 1)^2 \)

**Solution.** Here

\[ f(x, y) = (x - 2)^2 - y(y - 1)^2 \]
\[ \frac{\partial f}{\partial x} = 2(x-2), \quad \frac{\partial f}{\partial x} = -(y-1)^2 - 2y(y-1) \]
\[ = -(y-1)(3y-1) = -3y^2 + 4y - 1 \]
\[ = -(3y^2 - 3y + 1) = -(3y-1)(y-1) \]

\[ \frac{\partial^2 f}{\partial x^2} = 2, \]
\[ \frac{\partial^2 f}{\partial x \partial y} = 0, \]
\[ \frac{\partial^2 f}{\partial y^2} = -6y + 4 \]

The equations \( \frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0 \) on solving give \( x = 2, \quad y = 1 \) or \( \frac{1}{3} \).

The point \((2, 1)\) lies on the given curve, but \((2, \frac{1}{3})\) does not lie on the curve.

Step II. Now
\[ \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 - \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} \]
\[ = 0 - 2 (- 6y + 4) \]
\[ = 12y - 8. \]

Step III. At the point \((2, 1)\), this expression \(12y - 8\) is positive, hence there is a node at this point and there is no other double point on the given curve.

Solved Example 4. Determine the position and nature of the double points on the curve
\[ y(y - 6) = x^2(x - 2)^3 - 9 \]

Solution. Here
\[ f(x, y) = x^2(x - 2)^3 - y^2 + 6y - 9 \]

\[ \therefore \]
\[ \frac{\partial f}{\partial x} = 2x(x - 2)^3 + 3x^2(x - 2)^2 \]
\[ = x(x - 2)^3 [5x - 4] \]
\[ \frac{\partial f}{\partial y} = -2y + 6, \quad \frac{\partial^2 f}{\partial y^2} = -2 \]
\[ \frac{\partial^2 f}{\partial x^2} = (x - 2)^2 (5x - 4) + 2x(x - 2)(5x - 4) + 5x(x - 2)^2 \]
The equations \( \frac{\partial^2 f}{\partial x \partial y} = 0 \) are solved.

Solving the equations \( \frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0 \) we get

\[
\frac{\partial f}{\partial x} = x(x-2)^2(5x-4) = 0
\]

\[\Rightarrow\]

\[x = 0, 2, \frac{4}{5}\]

\[
\frac{\partial f}{\partial y} = -2y + 6 = 0
\]

\[\Rightarrow\]

\[y = 3\]

Hence we have to consider the points \((0, 3)\) \((2, 3)\) \((4, 3)\). Out of these points \((4, 3)\) does not satisfy the given equation.

\[\therefore\] \((0, 3)\) \((2, 3)\) are the double points,

Also \[
\left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 \left( -\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} \right) = 0 - (-16) (-2) = -32 \text{ i.e., negative at } (0, 3)
\]

\[\therefore\] There is a conjugate point at \((0, 3)\)

\[
\left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 \left( -\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} \right) = 0 - 0 (-2) = 0 \text{ at the point } (2, 3)
\]

\[\therefore\] There is a cusp at \((2, 3)\).

**Exercises**

1. Determine the position and nature of the double points on the curve 
   \[x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0.\]
2. Determine the position and nature of the double points on the curve. 
   \[x^4 - 2y^3 - 3y^2 - 2x^2 + 1 = 0.\]
3. Determine the position and nature of double points on the curve. 
   \[x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0\]
4. Determine the position anbd nature of the double points on the curve. 
   \[x^3 + y^3 = 3ax^2.\]
5. Determine the position and nature of the multiple points on the curve.

\[ x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0. \]

6. Find the tangents at the origin to the following curves:

(i) \[ a^2 \left( x^2 - y^2 \right) = x^2 y^2 \]

(ii) \[ \left( x^2 + y^2 \right)^2 = 4a^2 xy \]
**Introduction**

The object of curve tracing is to get an idea of the shape of the curve without undergoing the tedious process of plotting a large number of points close to each other. For this purpose the determination of the tangents, asymptotes and singular points is very important and useful. The tangent at a point gives the direction of the curve at the point, the asymptotes give an idea of the location of the curve at great distances from the origin and determination of the singular points gives an idea of the special form of the shape of the curve at these points. Maxima and minima determine the turning points on the curve.

Consideration of symmetry and knowledge of regions where branches of the curve do not exist are very important, because if one knows that the curve has symmetry in two regions, then the shape of the curve may be considered for one region only, and the part of the curve in the other region is obtained by symmetry. A branch of a curve does not exist in a certain region, if for a real value of one of the co-ordinates in this region, the other co-ordinate becomes imaginary.

We, therefore, proceed to consider symmetry for curves whose equations are given in cartesian co-ordinates. For other forms of the equations, this equation of symmetry will be considered at the appropriate place.

**1.1 Symmetry**

(i) A curve is symmetrical about the $x$-axis if corresponding to a point $(x, y)$ on the curve there is also the point $(x, y)$ lying on the curve. Evidently it would be so, if the equation of an algebraic curve when rationalized contains only even powers of $y$. For example consider the parabola $y^2 = 4ax$.

This curve is symmetrical about $x$-axis.

(ii) Similarly a curve is symmetrical about the $y$-axis if the equation of the algebraic curve when rationalised contains only even powers of $x$. e.g., $x^2 = 4by$.

The curve is symmetrical about the $y$-axis.

(iii) A curve is symmetrical about the line $y = x$, if the equation of the curve remains unchanged when $x$ and $y$ are interchanged, e.g., the curve $x^3 + y^3 = 3axy$ is symmetrical about $y = x$.

(iv) If corresponding to a point $P (x, y)$ on a curve there is also the point $P' (-x, -y)$ on it, there is symmetry about the origin or symmetry in opposite quadrants, meaning thereby that the curve when rotated through $180^\circ$ about the origin is again exactly in the original orientations, e.g., the rectangular hyperbola $xy = c^2$.

The curve is also symmetrical about the line $y = x$. 
1.2 Points on the Curve

Since in curve tracing, the co-ordinate axis are used, points of intersection of the curve with the axis must be found out and their nature studied. In particular, if the curve passes through the origin, the shape of the curve near the origin is easily found.

The intersections of the given curve with lines such as 

\[ y = \text{constant}, \ y = \pm x \] may also be studied with advantage.

The direction of the curve at any point is obtained by finding \( \frac{dy}{dx} \) from the equation of the curve \( f(x, y) = 0 \).

\[
\frac{dy}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial y}.
\]

At a point, where \( \frac{\partial f}{\partial x} = 0 \) but \( \frac{\partial f}{\partial y} \neq 0 \), the tangent to the curve is parallel to the \( x \)-axis while at a point, where \( \frac{\partial f}{\partial y} = 0 \) but \( \frac{\partial f}{\partial x} \neq 0 \), the tangent to the curve is parallel to the \( y \)-axis. The knowledge of such points makes the tracing more accurate.

The tangents at the origin in the case of an algebraic curve \( f(x, y) = 0 \) are, as shown earlier, given by equating to zero the sum of the lowest degree terms in \( f(x, y) \). In case the lowest degree terms are of the second or higher degree, the origin is a double point or a multiple point of higher order.

1.3 Region of non-existence

If possible, solve the equation \( f(x, y) = 0 \) for \( y \) in terms of \( x \) and study for what value of \( x \), the values of \( y \) are imaginary. Similarly solve for \( x \) in terms of \( y \) if possible, and examine when \( x \) is imaginary. The equation may be solved for other functions like \( x + y \) if convenient.

1.4 Asymptotes

Determine the asymptotes, if any, and find on which side of an asymptotes the curve lies. When the equation of the curve can be solved for \( x \) (or for \( y \)) the surest way of tracing it is to examine how it varies when the other variable tends from large negative values to large positive values.

1.5 Points of Inflexion

Definition: A point \( P \) on a curve is said to be a point of inflexion if the curve lies on opposite side of the tangent at \( P \) i.e., the curve crosses the tangent at \( P \). Points of inflexion are also examples of singular points on a curve.

In the below figure \( P \) is a point of inflexion, as the curve crosses the tangent at \( P \).

A point \( x = c \) is a point of inflexion on curve \( y = f(x) \) if \( f''(c) = 0 \) and \( f'''(c) \neq 0 \).

The position of the points of inflexion on a curve is independent of the choice of co-ordinate.
Hence a point $P$ is a point of inflexion on a curve $x = f(y)$.

\[ \frac{d^2x}{dy^2} = 0 \quad \text{and} \quad \frac{d^3x}{dy^3} \neq 0 \quad \text{at the point.} \]

**Example.** Here

\[ y = \frac{x^3}{a^2 + x^2} \]

\[ \frac{dy}{dx} = \frac{3x^2(a^2 + x^2) - 2x^3}{(a^2 + x^2)^2} = \frac{x^2(3a^2 + x^2)}{(a^2 + x^2)^2} \]

\[ \frac{d^2y}{dx^2} = \frac{(4x^3 + 6a^2x)(a^2 + x^2)^2 - 4x(3a^2 + x^2)(3a^2 + x^2)}{(a^2 + x^2)^4} \]

\[ = \frac{(4x^3 + 6a^2x)(a^2 + x^2)^2 - 4x(3a^2 + x^2)}{(a^2 + x^2)^4} \]

\[ = \frac{-2a^2x^3 + 6a^4x}{(a^2 + x^2)^3} \]

For points of inflexion

\[ \frac{d^2y}{dx^2} = 0 \]

\[ \frac{d^2y}{dx^2} = 0 \quad \Rightarrow \quad -2a^2x^3 + 6a^4x = 0 \]

\[ i.e., \quad -2a^2x(x^2 - 3a^2) = 0 \]

\[ \therefore \quad x = 0 \quad \text{or} \quad x = \pm \sqrt{3a} \]

You can easily verify that \( \frac{d^3y}{dx^3} \neq 0 \) at these points.

\[ \therefore \quad \text{The required points of inflexion are} \quad x = 0, \quad x = \pm \sqrt{3a} \]
In curve tracing sometimes the determination of the points of inflexion is helpful.

**Solved Example 1.** Trace the curve

\[ a^2 y^2 = x^2 (a^2 - x^2) \]

**Solution.** Since the equation of the curve of the curve contains only even powers of both \( x \) and \( y \), curve is symmetrical about both the axes of co-ordinates. We, therefore, need consider the curve only in the first quadrant, \( i.e., \) we consider the curve for positive values of \( x \) and \( y \).

The R.H.S. is negative if \( x > a \), hence no part of the curve lies to the right of the line \( x = a \).

The curve meets the \( x \)-axis at the points for which \( y = 0 \), \( i.e., \) where \( 0 = (a^2 - x^2)x^2 \), which gives \( x = 0 \) or \( x = \pm a \), \( i.e., \) the curve meets the \( x \)-axis at the points \((0, 0)\) and \((\pm a, 0)\).

Similarly, the point of intersection of the curve with the \( y \)-axis is the origin alone.

The tangents at the origin, obtained by equating to zero the lowest degree terms, are the lines \( a^2 (y^2 - x^2) \equiv 0 \), \( i.e., \) \( y^2 - x^2 = 0 \), which are two real and distinct lines \( y = \pm 1 \). Hence there is a node at the origin.

Differentiating \( w.r.t. \) \( x \) the equation of the curve, we obtain

\[
\frac{dy}{dx} = \frac{x}{a^2 y} (a^2 - 2x^2) \]

\[
= \frac{a^2 - 2x^2}{a\sqrt{a^2 - x^2}}
\]

At \((a, 0)\), \( \frac{dx}{dy} = 0 \).

i.e., the tangent at \((a, 0)\) is parallel to the \( y \)-axis.
Also
\[ \frac{dy}{dx} = 0, \text{ when } x = \pm \frac{a}{\sqrt{2}} \]

and \[ \therefore \quad y = \pm \frac{x}{a} \sqrt{a^2 - x^2} = \pm \frac{a}{2} \]

\[ \therefore \quad \text{The tangent is parallel to the } x\text{-axis at} \left( \frac{\mp a}{\sqrt{2}}, \frac{\pm a}{2} \right) \]

There is no asymptote, from the above considerations the curve is as shown in the above diagram.

One could also conclude that the maximum value of \( x^2 (a^2 - x^2) \) is \( \frac{1}{4} a^4 \). Hence no value of \( y \) on the curve is numerically greater than \( \frac{1}{2} a \).

**Solved Example 2.** Trace the cubical parabola \( a^2 y = x^3 \).

**Solution.**

(i) Here the curve passes through the origin and the tangent at the origin is \( y = 0 \), *i.e.*, the \( x\)-axis.

(ii) There is a symmetry in opposite quadrants, as corresponding to a point \( (x, y) \) on the curve, there is a point \( (-x, -y) \) on the curve.

Also when \( x \) is positive \( y \) is also positive and when \( x \) is a negative, \( y \) is at the same time negative. Thus the curve exists only in the first and the third quadrants. When \( x \to \infty \), \( y \to \infty \).

(iii) There is no asymptote to the curve.

![Fig. 2](image-url)
There is a point of inflexion at the origin as 
\[
\frac{d^2y}{dx^2} = \frac{6x}{a^2}
\]

is zero when \( x = 0 \), and \( \frac{d^3y}{dx^3} = \frac{6}{a^2} \) at the origin is different from zero.

**Solved Example 3.** Trace \( ay^2 = x^3 \), the semi-cubical parabola.

**Solution.**

(i) Here the curve passes through the origin, the tangents at the origin being \( y^2 = 0 \), i.e., two coincident lines \( y = 0 \). Hence the origin is a cusp and the \( x \)-axis is a tangent at the cusp.

(ii) Assume \( a > 0 \). The curve is symmetrical only about the \( x \)-axis as the equation contains only even powers of \( y \).

(iii) The curve exists for positive values of \( x \) only, since
\[
y = \pm x \sqrt{\frac{x}{a}}.
\]

(iv) As the curve exists for positive values of \( x \) only and also the curve is symmetrical about the \( x \)-axis, there is a cusp at the origin.

(v) As \( x \) increases through positive values, \( y \) also increases numerically.

(vi) There is no asymptote of the curve and there is no point of inflexion on the curve.

**Solved Example 4.** Trace \( ay^2 = x^2 (x - a) \).

**Solution.**

(i) The curve is symmetrical about the \( x \)-axis as terms involving even powers of \( y \) only occur in the equation.

(ii) The curve passes through the origin, the tangents at the origin being given by \( a(x^2 + y^2) = 0 \) which represents two imaginary lines \( y = \pm ix \). Hence the origin is a conjugate point (i.e., an isolated point) on the curve.
Here
\[ y = \pm x \sqrt{\frac{x-a}{a}} \]

(iii) The curve does not exist for negative values of \( x \), as \( y \) becomes imaginary. Also the curve does not exist for positive values of \( x \) lying in the region \( 0 < x < a \). The curve exists for values of \( x \geq a \).

(iv) The curve cuts the \( x \)-axis at the point \( (a, 0) \). Shifting the origin to the point \( (a, 0) \), we find that the equation of the curve becomes \( ay^2 = (x + a)^2 \) and hence at the new origin the equation of the tangent to the curve is \( x = 0 \), i.e., at the given point \( (a, 0) \) the equation of the tangent to the given curve is \( x = a \) with reference to the given axes of co-ordinates.

(v) As \( x \) increases, for values \( x > a \), \( y \) also increases.

(vi) There is no asymptote of the curve.

(vii) There is a point of inflexion at \( x = \frac{4a}{3} \), given by
\[ \frac{d^2y}{dx^2} = 0 \quad \text{and} \quad \frac{d^3y}{dx^3} \neq 0. \]

Solved Example 5. Trace \( x (x^2 + y^2) = a (x^2 - y^2) \).

Solution.

(i) As only even powers of \( y \) occur in the given equation the curve is symmetrical about the \( x \)-axis only.

(ii) The curve passes through the origin, the equation of the tangents at the origin being \( x^2 - y^2 = 0 \), which are two real and distinct lines \( y = \pm x \). Hence the origin is node on the curve.
(iii) Rewriting the equation in the form
\[ 0, y^3 + y^2 (a + x) + x^3 - ax^2 = 0, \]
we find that [since there is no term involving \( y^3 \), and the co-efficient of \( y^2 \) is a linear function of \( x \), viz., \( a + 2 \)] there is the asymptote \( x = -a \); which is parallel to the \( y \)-axis. Satisfy yourself that is no other asymptote.

Fig. 5

(iv) The curve cuts the \( x \)-axis at \( x = 0 \) and \( x = a \).

(v) Rewriting the equation in the form \( y = \pm x \sqrt{\frac{a-x}{a+x}} \)

We find that \( y \) is imaginary when \( x > a \), \( \therefore \) no part of the curve exists to the right hand side of the line \( x = a \). Also \( x \) is imaginary when \( x < -a \) hence no part of the curve exists to the left hand side of the line \( x = -a \). Also the tangent at \( (a, 0) \) is parallel to the \( y \)-axis.

Solved Example 6. Trace the curve
\[ x = a (\theta - \sin \theta), \quad y = a (1 - \cos \theta) \]

Solution. Here the equations of the curve are written in parametric form, where \( \theta \) is the parameter. If \( \theta \) is replaced by \(-\theta\), \( y \) is unaltered while the numerical value of \( x \) is changed in sign, so the curve is symmetrical about the \( y \)-axis.

\[ \frac{dy}{dx} = \frac{dy}{d\theta} \frac{d\theta}{dx} = \frac{\sin \theta}{1 - \cos \theta} = \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}} = \cot \frac{\theta}{2} \]

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i.e., \[ \tan \psi = \cot \frac{\theta}{2} = \tan \left( \frac{\pi - \theta}{2} \right) . \]

whence \[ \psi = \frac{\pi - \theta}{2} . \]

Giving for the parameter \( \theta \), a few values, the values of \( x, y \) and \( \psi \) are shown below:

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( x )</th>
<th>( y )</th>
<th>( \psi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\pi)</td>
<td>(-a\pi)</td>
<td>(2a)</td>
<td>(\pi)</td>
</tr>
<tr>
<td>(-\frac{\pi}{2})</td>
<td>(\left(\frac{-\pi}{2} + 1\right)a)</td>
<td>(a)</td>
<td>(\frac{3\pi}{4})</td>
</tr>
<tr>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(\frac{\pi}{2})</td>
</tr>
<tr>
<td>(\frac{\pi}{2})</td>
<td>(a\left(\frac{\pi}{2} - 1\right))</td>
<td>(a)</td>
<td>(\frac{\pi}{4})</td>
</tr>
<tr>
<td>(\pi)</td>
<td>(a\pi)</td>
<td>(2a)</td>
<td>(0)</td>
</tr>
<tr>
<td>(\frac{3\pi}{2})</td>
<td>(a\left(\frac{3\pi}{2} + 1\right))</td>
<td>(a)</td>
<td>(-\frac{\pi}{4})</td>
</tr>
<tr>
<td>(2\pi)</td>
<td>(2\pi a)</td>
<td>(0)</td>
<td>(-\frac{\pi}{2})</td>
</tr>
<tr>
<td>(\frac{5\pi}{2})</td>
<td>(\left(\frac{5\pi}{2} - 1\right))</td>
<td>(a)</td>
<td>(-\frac{3\pi}{4})</td>
</tr>
<tr>
<td>(3\pi)</td>
<td>(3\pi a)</td>
<td>(2a)</td>
<td>(-\pi)</td>
</tr>
</tbody>
</table>

Since \(-1 \leq \cos \theta \leq 1\), \(y\) lies between 0 and \(2a\). As \(\theta \to \pm \infty\), \(x \to \pm \infty\), but \(y\) does not tend to any limit, there is no-asymptote, although the curve extends from \(-\infty\) to \(+\infty\).

This curve is known as the cycloid. It consists of an unlimited number of arches like \(A_1B_1O, OB_2A_2, A_2B_3A_3\) etc. The name cycloid is generally given to any one of these complete arches. If we consider the arch \(OB_2A_2\), its equations in parametric form are given by

\[
\begin{align*}
x &= a \left( \theta - \sin \theta \right), \\
y &= a \left( 1 - \cos \theta \right),
\end{align*}
\]

where \(\theta\) varies from 0 to \(2\pi\). The length of \(OCA_2\) along the \(x\)-axis is called the base of the cycloidal arch \(OB_2A_2\), the length of this base is \(2\pi a\). The highest ordinate of a point on the arch \(OB_2A_2\), occurs at \(B_2\). The co-ordinates of \(B_2\) are \((\pi a, 2a)\) corresponding to the value of the parameter \(\theta = \pi\). The point \(B_2\) is called the vertex of the cycloid. There are cusps at \(O_2, A_2\) for the arch \(OB_2A_2\). The tangent at \(B_2\) is parallel to the \(x\)-axis as at \((a\pi, 2a), \psi = 0\). The tangents at the cusps \(O, A_2\) (of the arch \(OB_2A_2\)) are parallel to the \(y\)-axis.
Fig. 6

\[ x = a (\theta + \sin \theta) \quad \text{and} \quad y = a (1 + \cos \theta) \]

Solved Example 7. Trace the curve \( x^{2/3} + y^{2/3} = a^{2/3} \).

Solution. Let us express the equation of the curve in a parametric form and then trace it. The equation can be written as

\[
\left( \frac{x}{a} \right)^{2/3} + \left( \frac{y}{a} \right)^{2/3} = 1.
\]

Put

\[
\left( \frac{x}{a} \right)^{1/3} = \cos \theta, \quad \text{and} \quad \left( \frac{y}{a} \right)^{1/3} = \sin \theta.
\]

so that

\[ x = a \cos^3 \theta, \quad y = a \sin^3 \theta \]

are the parametric equations of the given curve.

Since \( \cos \theta \) and \( \sin \theta \) cannot exceed 1 numerically, no point of the curve has its abscissa and the ordinate greater than \( a \) numerically. The curve is therefore bounded.

When \( \theta \) is replaced by \( -\theta \), \( \cos \theta \) does not change but \( \sin \theta \) changes in sign. Thus when \( \theta \) is changed to \( -\theta \) since \( x \) does not change and \( y \) is changed in sign only, the curve is symmetrical about the \( x \)-axis.

Similarly when \( \theta \) is changed to \( \pi - \theta \), \( \sin \theta \) does not change but \( \cos \theta \) change in sign. Thus when
\( \theta \) is changed to \( \pi - \theta \), since \( y \) does not change and \( x \) is changed in sign only, the curve is symmetrical about the \( y \)-axis as well.

**Note:** \( \theta \) is a parameter here and *not the vectorial angle of a point*. As \( \theta \) changes from 0 to \( \frac{\pi}{2} \), both \( x \) and \( y \) remain positive and to get the shape of the curve in the first quadrant, \( \theta \) must vary from 0 to \( \frac{\pi}{2} \).

Also as \( \theta \) changes from 0 to \( \frac{\pi}{2} \), \( x (= a \cos \theta) \) must diminish from \( a \) to 0 and \( y (= a \sin \theta) \) must increase from 0 to \( a \).

Now

\[
\frac{dy}{dx} \frac{d}{d\theta} = \frac{3a\sin^2 \theta \cos \theta}{3a \cos^3 \theta (-\sin \theta)} = -\tan \theta
\]

*i.e.,*

\[
\tan \psi = -\tan \theta = \tan (\pi - \theta)
\]

or

\[
\psi = \pi - \theta.
\]

.: When

\[
\theta = 0, \quad \psi = \pi.
\]

Let us draw the curve in the first quadrant. This portion of the curve in the first quadrant corresponds to \( \theta \) varying from 0 to \( \frac{\pi}{2} \). Since the curve is symmetrical with respect to \( x \)-axis, the portion of the curve for the 4th quadrant is drawn from its shape in the first quadrant. Since the curve is also symmetrical with respect to the \( y \)-axis, the portion of the curve for the 2nd quadrant is drawn, from its shape in the first quadrant. After the curve for the 2nd quadrant is drawn, the curve for the 3rd quadrant can be drawn from symmetry with respect to \( x \)-axis. Thus knowledge of the shape of the curve in the first quadrant only, enables us to draw the curve for all the four quadrants.

**Fig. 7**
At $A_1$ there is cusp $\psi$ at this point being given by $\psi = \pi$, the tangent at $A_1$ being the $x$-axis. Similarly there are cusps at $B_1, A_2$ and $B_2$. Now $\psi$ at $B_1$ is $\frac{\pi}{2}$ and the tangent at $B_1$ is the $y$-axis.

**Exercises**

Trace the following curves:

1. $y^2 (2a - x) = x^2$
   
   *[Hind: This curve is symmetrical about the $x$-axis. The curve does not exist for negative values of $x$. Origin is a cusp on the curve, the $x$-axis being the tangent at the cusp. There is the asymptote $2a - x = 0$. The curve does not exist on the right-hand side of the line $x = 2a$.]*

![Fig. 8](image1)

2. $xy^2 = 4a^2 (2a - x)$

   *[Hint: Curve is symmetrical about the $x$-axis, it does not pass through the origin. Its asymptote is $x = 0$. The curve does not exist for negative values of $x$ or positive values of $x$ greater than $2a$.]*

![Fig. 9](image2)
The curve cuts the $x$-axis at $(2a, 0)$ and the tangent to the curve at $(2a, 0)$ is parallel to the $y$-axis.
There are two points of inflexion. Find these points.]
3. $ay^2 = x^2 (a - x)$.
4. $9ay^2 = x (x - 3a)^2$.
5. $x^3 + y^3 - 3axy = 0$.
[Hint: The curve is symmetrical about the line $y = x$. The line $y = x$ meets the curve at $(0, 0)$ and also at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$]

Fig. 10

The curve passes through the origin and $x = 0, y = 0$ are the tangents there, so that origin is a node on the curve. $x + y + a = 0$ is the only asymptote.]
6. $y^2 (a^2 + x^2) = x^2 (a^2 - x^2)$
[Hint: It is symmetrical about both the axes. It passes through the origin and $y = \pm x$ are the two tangents there. So that the origin is a node on the curve. It has no asymptote, $y$ is real only when $x$ lies between $-a$ and $a$.]

Fig. 11
7. \( y^2 (a + x) = x^2 (3a - x) \).

[Hint: The curve is symmetrical about the \( x \)-axis and lies between \( x = -a \) and \( x = 3a \). \( x = -a \) is the asymptote.]

![Fig. 12](image)

Origin lies on the curve and \( y = \pm \sqrt{3x} \) are the two tangents at the origin \( \therefore \) origin is a node on the curve.]

8. \( y^2 (a^2 - x^2) = x^4 \).

![Fig. 13](image)
[**Hint:** Curve is symmetrical about both the axes. 

\(x = \pm a\) are the asymptotes. No value of \(x\) can numerically exceed \(a\). The \(x\)-axis is tangent at the origin.]

9. \(ay^2 = x (x^2 + y^2) \ (a > 0)\).
LESSON 7

CURVE TRACING – II (POLAR CO-ORDINATE)

2. Polar Equations

The Polar equations can be transformed into cartesian equations and the curve represented by it can then be traced by rules already explained. But it is often more convenient to trace the curve directly from its polar equation. For this purpose the rules for symmetry are the following:

Let \( f(r, \theta) = 0 \) be the equation of a curve.

When \( \theta \) is changed to \(-\theta\), if the equation remains unchanged then the curve is \textit{symmetrical about the line} \( \theta = 0 \), \textit{i.e.}, the initial line. The curves.

\[
\begin{align*}
  r^2 &= a^2 \cos \theta, \\
  r &= a \cos \theta, \\
  r^2 &= \frac{a^2 \cos 2\theta}{\cos^2 \theta + \sin^2 \theta}
\end{align*}
\]

are all symmetrical about the initial line.

In general if \( f(r, \theta) = f(r, 2\alpha - \theta) \) then the curve is symmetrical about the line \( \theta = 2\alpha - \theta \), \textit{i.e.}, the line \( \theta = \alpha \).

If when \( \theta \) is changed to \( \theta + \pi \), the equation of the curve is unchanged, the curve has symmetry in opposite quadrants

\( \text{e.g.,} \quad r^2 = a^2 \cos^2 \theta + b^2 \sin \theta. \)

If \( r \) is expressed in terms of \( \sin \theta \) and \( \cos \theta \) only, the same value of \( r \) is obtained when the angle is increased by \( 2\pi \). In such cases it is sufficient to consider the value of \( \theta \) in the range of width \( 2\pi \), \textit{i.e.}, from 0 to \( 2\pi \) or from \(-\pi \) to \( \pi \).

If \( r \to \infty \) when, \( \theta \to \alpha \), then examine whether the, curve has an asymptote parallel to the line \( \theta = \alpha \).

The direction of the tangent at any point \((r, \theta)\) on the curve can be found from the value of

\[
\tan \phi = r \frac{d\theta}{dr}
\]

\textbf{[Note:} ‘\( \phi \)’ is the angle between the radius vector at a point and the tangent to the curve at the point.\]

By giving different values to \( \theta \) we can find the corresponding values of \( r \) and the slopes of the tangents there, can be tabulated. Sometimes it is inconvenient to find the corresponding value of \( r \) for certain value of \( \theta \) and even if we find they involve radical, then in such cases we should consider a particular region for \( \theta \) and ascertain whether \( r \) increases or decreases in that region.\]

\textit{e.g.,} \( r = a (1 + \cos \theta) \)
Here we did not find the value of \( r \) corresponding to \( q = \frac{\pi}{6} \). But we may take into account that as \( q \) increases from 0 to \( \frac{\pi}{2} \), \( \cos q \) will go on decreasing and consequently \( r = a \left(1 + \cos q\right) \) will go on decreasing. Again we have not found the value of \( r \) corresponding to \( q = \frac{5\pi}{6} \). But we may take into account that as \( q \) increases from \( \frac{\pi}{2} \) to \( \pi \), \( \cos q \) will increase in magnitude but will be negative being in 2nd quadrant and consequently \( r \) will go on decreasing, continuously.

No part of the curve shall exist for those values of \( q \) which make corresponding value of \( r \) imaginary.

We also find the limits to the value of \( r \).

\[
e.g., \quad r = a \sin 2q.
\]

Now whatever \( \theta \) may be, \( \sin 2\theta \) can never be greater than unity and hence \( a \sin 2\theta \) or \( r \) shall never be greater than \( a \), \( i.e. \), the curve shall entirely lie within the circle of radius \( a \).

**Solved Example 1.** Trace the cardioid \( r = a \left(1 + \cos \theta\right) \).

**Solution.** The equation remains unaltered by putting \(-d\) for \(d\). Hence the curve is symmetrical about the initial line.

Since \(-1 \leq \cos \theta \leq 1\) the values of \( r \) lie between 0 and \( 2a \).

Also logarithmic differentiation gives

\[
\frac{1}{2} \frac{dr}{d\theta} = -\sin \alpha = -\frac{2\sin \frac{\theta}{2} \cos \frac{\theta}{2}}{\cos \frac{\theta}{2}} \quad \frac{\theta}{2}
\]

or

\[
\frac{1}{r} \frac{d\theta}{dr} = -\tan \frac{\theta}{2}
\]

\[i.e.,\]

\[
\cot \phi = -\tan \frac{\theta}{2} = \cot \left(\frac{\pi}{2} + \frac{\theta}{2}\right)
\]

\[
\therefore \quad \phi = \frac{\pi}{2} + \frac{\theta}{2}
\]

Tabulating \( r \) and \( \phi \) for a few values of \( \theta \) from 0 to \( \pi \), we get

\[
\begin{array}{c|c|c|c|c|c}
\theta & 0 & \frac{\pi}{3} & \frac{\pi}{2} & \frac{2\pi}{3} & \pi \\
r & 2a & \frac{3a}{2} & a & \frac{a}{2} & 0 \\
\end{array}
\]
### Table

<table>
<thead>
<tr>
<th>$0$</th>
<th>$r$</th>
<th>$\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$2a$</td>
<td>$\pi/2$</td>
</tr>
<tr>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{a(1+\sqrt{3})}{2}$</td>
<td>$\frac{7\pi}{12}$</td>
</tr>
<tr>
<td>$\frac{\pi}{3}$</td>
<td>$\frac{3a}{2}$</td>
<td>$\frac{2\pi}{3}$</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>$a$</td>
<td>$\frac{3\pi}{4}$</td>
</tr>
<tr>
<td>$\frac{2\pi}{3}$</td>
<td>$\frac{a}{2}$</td>
<td>$\frac{5\pi}{6}$</td>
</tr>
<tr>
<td>$\pi$</td>
<td>$0$</td>
<td>$\pi$</td>
</tr>
</tbody>
</table>

Plotting and marking out the directions of the tangents, the shape of the curve is as shown.

Note: $\phi$ at A is $\pi/2$ i.e., Tangent at A is perpendicular to x-axis. Again $\phi$ at O, where $r = 0$ is $\pi$. The point O is a cusp, the tangent at O being the initial line.

### 3. Tracing of Polar Curves

In tracing the curves of the type $r = a \sin n\theta$ or $r = a \cos n\theta$. We divide each quadrant into $n$ equal parts and give the values to $\theta$ and find the corresponding values of $r$. The points thus plotted will give the shape of the curve. It will be observed from the following two examples that above types of curves will consist either $n$ or $2n$ equal loops according as $n$ is odd or even.
Solved Example 2. Trace the curve \( r = a \cos \theta \).

Solution. Here we shall divide each quadrant into two equal parts, i.e., first quadrant is divided into two parts 0 to \( \frac{\pi}{4} \) and \( \frac{\pi}{4} \) to \( \frac{\pi}{2} \) and so on.

The equation remains unaltered by putting \(-\theta\) for \( \theta \). Hence the curve is symmetrical about the initial line.

![Fig. 2](image)

Tabulating \( r \) for a few values of \( \theta \) from 0 to \( 2\pi \), we get

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( 2\theta )</th>
<th>( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>( a )</td>
</tr>
<tr>
<td>( \frac{\pi}{4} )</td>
<td>( \frac{\pi}{2} )</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>( \pi )</td>
<td>(-a)</td>
</tr>
<tr>
<td>( \frac{3\pi}{4} )</td>
<td>( \frac{3\pi}{2} )</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{5\pi}{4} )</td>
<td>( \frac{5\pi}{2} )</td>
<td>( a )</td>
</tr>
<tr>
<td>( \pi )</td>
<td>( 2\pi )</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{7\pi}{4} )</td>
<td>( \frac{7\pi}{2} )</td>
<td>(-a)</td>
</tr>
<tr>
<td>2\pi</td>
<td>4\pi</td>
<td>( a )</td>
</tr>
</tbody>
</table>
Plotting these points and we find that there are 2, i.e., 4 equal loops. **Note:** here \( n = 2 \).

**Solved Example 3.** Trace the curve \( r = a \sin \vartheta \).

**Solution.** The equation remains unaltered on putting for \( \vartheta \) either

\[
\frac{\pi}{3} - \theta \text{ or } \pi - \theta \text{ or } \frac{5\pi}{3} - \theta \text{ or } \frac{7\pi}{3} - \theta.
\]

Hence the curve is symmetrical about the lines \( \vartheta = \pi / 3 - \theta \) or \( \pi - \theta \) etc.

* i.e., about the line \( \vartheta = \frac{\pi}{6}, \vartheta = \frac{\pi}{2}, \vartheta = \frac{5\pi}{6} \) etc.

The lines \( \vartheta = \frac{7\pi}{6}, \vartheta = \frac{9\pi}{6} \) are repetitions of the lines \( \vartheta = \frac{\pi}{6}, \vartheta = \frac{\pi}{2} \) etc.

Since \( \sin \vartheta \) lies between \(-1\) and \(+1\), \( r \) cannot exceed \( a \) numerically, so the curve is bounded and there is no asymptote.

Here we shall divide each quadrant into three equal parts, i.e., first quadrant is divided into three parts \( 0 \) to \( \frac{\pi}{6} \), \( \frac{\pi}{6} \) to \( \frac{\pi}{3} \) and \( \frac{\pi}{3} \) to \( \frac{\pi}{2} \).

A few sets of values of \( \vartheta \), and \( r \) are tabulated below:

<table>
<thead>
<tr>
<th>( \vartheta )</th>
<th>( 3\vartheta )</th>
<th>( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{\pi}{6} )</td>
<td>( \frac{\pi}{2} )</td>
<td>( a )</td>
</tr>
<tr>
<td>( \frac{\pi}{3} )</td>
<td>( \pi )</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>( \frac{3\pi}{2} )</td>
<td>(-a)</td>
</tr>
<tr>
<td>( \frac{2\pi}{3} )</td>
<td>( 2\pi )</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{5\pi}{6} )</td>
<td>( \frac{5\pi}{2} )</td>
<td>( a )</td>
</tr>
<tr>
<td>( \pi )</td>
<td>( 3\pi )</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{7\pi}{6} )</td>
<td>( \frac{7\pi}{2} )</td>
<td>(-a)</td>
</tr>
</tbody>
</table>
Here we have stopped at $\theta = \frac{7\pi}{6}$. The reason is that when $\theta = \frac{7\pi}{6}$, $r = -a$ and we get the point $\left(-a, \frac{7\pi}{6}\right)$ which is the same as the point $\left(a, \frac{\pi}{6}\right)$, which we have already found. Hence by giving to $\theta$ values greater than $\pi$, no new points will be obtained but the same old points shall be repeated.

These points when plotted enable the loops to be traced. On account of symmetry about the line $\theta = \frac{\pi}{2}$ from loop No. I we get loop No. II lying between the lines $\theta = \frac{2\pi}{3}$ and $\theta = \pi$. Finally from loop No. I on account of symmetry about the line $\theta = \frac{5\pi}{6}$ we get loop No. III between the lines $\theta = \frac{4\pi}{3}$ and $\theta = \frac{5\pi}{3}$.

In general the curve $r = a \sin (2m + 1) \theta$ looks like $2n + 1$ leaved flower. On the contrary the curve $r = a \sin 2m\theta$ looks like flower having $4n$ leaves, where $m$ is odd or even.

The curve $r = a \sin 3\theta$ has three loops and is known as a 3 leaved rose.

**Solved Example 4.** Trace the curve (Leminscate of Bernouilli)

$$r^2 = a^2 \cos 2\theta.$$

**Solution.** The equation remains unaltered by putting $-\theta$ for $\theta$. Hence the curve is symmetrical about the initial line $\theta = 0$.

A few sets of values of $r$ and $\theta$ are tabulated below:
<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$2\theta$</th>
<th>$r^2$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$a^2$</td>
<td>$a$</td>
</tr>
<tr>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\pi}{2}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>$\pi$</td>
<td>$a^2$</td>
<td>imaginary</td>
</tr>
<tr>
<td>$\frac{3\pi}{4}$</td>
<td>$\frac{3\pi}{2}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\pi$</td>
<td>$2\pi$</td>
<td>$a^2$</td>
<td>$a$</td>
</tr>
</tbody>
</table>

These points when plotted enable the loops to be traced. We find that when $\theta$ increases from 0 to $\frac{\pi}{4}$, $r$ diminishes from $a$ to 0.

As $\theta$ increases from $\frac{\pi}{4}$ to $\frac{3\pi}{4}$, $r$ is imaginary.

And as $\theta$ increases from $\frac{3\pi}{4}$ to $\pi$, $r$ increases from 0 to $a$.

$\therefore$ The curve consists of two loops between the lines $\theta = \frac{\pi}{4}$, $\theta = \frac{3\pi}{4}$.
**Solved Example 5.** Trace the curve $r^m = a^m \cos m\theta$ for $m = 1, -1, 2, -2, \frac{1}{2}, -\frac{1}{2}$.

**Solution.**

(i) For $m = 1$, we have

\[
\begin{align*}
  r &= a \cos \theta \\
  r^2 &= ar \cos \theta \\
  x^2 + y^2 &= ax
\end{align*}
\]

which is a circle (see fig. 5) with its centre at $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$.

(ii) For $m = -1$, we have

\[
\frac{1}{r} = \frac{1}{a} \cos (-b)
\]

\[
\Rightarrow \quad a = r \cos \theta
\]

\[
\Rightarrow \quad x = a
\]

which is a straight line perpendicular to the initial line and at a distance $a$, from it.
(iii) For $m = 2$, we have
\[ r^2 = a^2 \cos 2\theta \]
which is lemniscate of Bernoulli. (See solved example 4)
(iv) For $m = -2$, we have
\[ r^2 = a^{-2} \cos (-2\theta) \]
\[ \Rightarrow \quad r^2 \cos 2\theta = a^2 \]
\[ \Rightarrow \quad r^2 (\cos^2 \theta - \sin^2 \theta) = a^2 \]
\[ \Rightarrow \quad x^2 - y^2 = a^2 \]
which is known to be a rectangular hyperbola (Fig. 7).

To trace the curve write its equation in the form
\[ r^2 = \frac{a^2}{\cos 2\theta} \]
The curve is symmetrical about the initial line.
The following table gives the corresponding variation in $\theta$ and $r$ only.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$2\theta$</th>
<th>$\cos 2\theta$</th>
<th>$r^2$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>$1$</td>
<td>$a^2$</td>
<td>$\pm a$</td>
</tr>
<tr>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\pi}{2}$</td>
<td>$0$</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>$\pi$</td>
<td>$-1$</td>
<td>$-a^2$</td>
<td>Imaginary</td>
</tr>
<tr>
<td>$\frac{3\pi}{2}$</td>
<td>$\frac{3\pi}{4}$</td>
<td>$0$</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$\pi$</td>
<td>$2\pi$</td>
<td>$1$</td>
<td>$a^2$</td>
<td>$\pm a$</td>
</tr>
</tbody>
</table>
[Note: The lines \( y = x, y = -x \) are the asymptotes of the rectangular hyperbola and they are inclined at angles \( \pm \frac{\pi}{4} \) to the axis of \( x \), and hence are at right angles.]

\[
\Rightarrow \quad r = a \cos^2 \left( \frac{1}{2} \theta \right) \\
\Rightarrow \quad 2r = a \left( 1 + \cos \theta \right)
\]

which is a cardioid.

(vi) For \( m - \frac{1}{2} \), we have

\[
r^{\frac{1}{2}} = a^{\frac{1}{2}} \cos \left( \frac{1}{2} \theta \right)
\]

\[
\Rightarrow \quad a^2 = r^2 \cos \frac{1}{2} \theta
\]

\[
\Rightarrow \quad 2a = r \left( 1 + \cos \theta \right)
\]

\[
\Rightarrow \quad \frac{2a}{r} = (1 + \cos \theta)
\]

which is known to be a parabola (Fig. 8).

To trace the curve we rewrite the equation in the form

\[
r = \frac{2a}{1 + \cos \theta}
\]

The curve is symmetrical about the initial line. The following table gives the corresponding variations in \( \theta \) and \( r \).
Exercises

Trace the following curves:

(i) \( r = a (2 \cos \theta + \cos 3\theta) \)

(ii) \( r = a \cos 3\theta \)

(iii) \( r = a \sin 2\theta \)

[Hint: We will get four loops corresponding to the variation of \( \theta \) in the intervals
\[ \left[ 0, \frac{\pi}{2} \right], \left[ \frac{\pi}{2}, \pi \right], \left[ \pi, \frac{3\pi}{2} \right], \left[ \frac{3\pi}{2}, 2\pi \right] \] respectively.]

(iv) \( x^3 + y^3 = 3axy \)

[Hint: Put \( x = r \cos \theta, y = r \sin \theta \) then equation of curve is transformed into
\[ r = \frac{3a \sin \theta \cos \theta}{\cos^3 \theta + \sin^3 \theta} = \frac{3a \tan \theta \sec \theta}{1 + \tan^3 \theta} \]

(v) \( r = a \sin 2\theta \)

Ans.
(vi) $r = a (1 + \cos \theta)$

Ans.
Exercise

1. Find the equation of the tangent and normal the curve \( x = a \cos \theta, \ y = b \sin \theta \) at any point ‘\( \theta \)’.

2. Find the tangent and normal to the curve:
   
   (i) \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) at the point \((a, 0)\).
   
   (ii) \( x = a (\theta + \sin \theta), \ y = a (1 + \cos \theta) \) at \( \theta = \frac{\pi}{2} \).
   
   (iii) \( y = c \cos \frac{x}{c} \) at the point \((0, c)\).

3. Show the line \( x \cos^3 \theta + y \sin^3 \theta = c \) is a tangent to the curve \( x^2y^2 = a^2 (x^2 + y^2) \).
   
   [Hint: The equation of the curve can be written as \( \frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{c^2} \) ]

4. Prove that the straight line \( \frac{x}{a} + \frac{y}{b} = 2 \) touches the curve \( \left( \frac{x}{a} \right)^n + \left( \frac{y}{b} \right)^n = 2 \) at the point \((a, b)\) whatever be the value of \( n \).

5. At what point of the curve \( y = x^2 - 3x + 2 \) is the tangent perpendicular to the line \( y = x ? \)
   
   [Ans. \((1, 0)\)]

6. At what point of the curve \( y = 2x^3 + 3x^2 - 10x + 7 \) are the tangents parallel to the line \( y = 2x ? \)
   
   [Ans. \((1, 2), (-2, 23)\)]
Exercise

1. Prove that the equation of the tangent at any point \((4m^2, 8m^3)\) of the semicubical parabola \(x^3 - y^3 = 0\) is \(y = 3mx - 4m^3\) and show that it meets the curve again at \((m^2, -m^3)\), where it is normal if \(9m^2 = 2\).

2. Show that the normal at any point of the curve \(x = a \cos \theta = a \sin \theta\), \(y = a \sin \theta - a \cos \theta\) is at a constant distance from the origin.

3. The tangent at any point on the curve \(x^3 + y^3 = 2a^3\) cuts off lengths \(p\) and \(q\) on the co-ordinate axis, show that \(p^{3/2} + q^{3/2} = 2^{1/2}a^{3/2}\).
Exercise

1. Find the angle between the following pairs of curves at each one of their points of intersection

   (i) \( x^2 - y^2 = a^2, x^2 + y^2 = \sqrt{2}a^2 \).

   (ii) \( x^2 - y^2 = 8, xy = 3 \).

   (iii) Prove that the curves \( y = 1 - ax^2 \) and \( y = x^2 \) cut orthogonally when \( a = \frac{1}{3} \).

   (iv) Prove that the curves:

   \[
   \begin{align*}
   x^2 + 2xy - y^2 + 2ax &= 0 \\
   3y^3 - 2a^2x - 4a^2y + a^3 &= 0
   \end{align*}
   \]

   intersect at an angle \( \tan^{-1}\left(\frac{9}{8}\right) \) at the point \((a, -a)\).
Exercise

1. Find the lengths of the subtangent, subnormal, tangent and normal for the following curves:
   (i) \( 2x^2 - 3y^2 = 5 \) at (3, 1)
   (ii) \( x = a \cos^3 \theta, y = a \sin^3 \theta \) at \( \theta \)
   (iii) \( x = a (\theta - \sin \theta), y = a (1 - \cos \theta) \) at \( \theta = \frac{\pi}{2} \)

2. Find the lengths of the normal and subnormal to the curve:
   \[ y = \frac{a}{2} \left[ e^{x/a} + e^{-x/a} \right] \]

3. Show that the subtangent at any point of the curve \( x^m y^n = a^{m+n} \) varies as the abscissa.
4. Show that in the parabolas \( y^2 = 4ax \), the subnormal is constant and the subtangent varies as the abscissa of the point of contact.
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Exercise

1. Prove that in the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), the length of the normal varies inversely as the perpendicular from the origin upon the tangent.

2. For the catenary \( y = c \cos \left( \frac{x}{c} \right) \), prove that the length of the normal is \( \frac{y^2}{c} \).

3. Show that the subnormal at any point of the curve \( y^2x^2 = a^2(x^2 - a^2) \) varies inversely as the cube of abscissa \( a \).

4. Show that for the curve \( by^2 = (x + a)^3 \), the square of the subtangent varies as the subnormal.

5. Show that in the curve \( y = a \log(x^2 - a^2) \), the sum of the tangent and the subtangent varies as the product of the co-ordinates of the point.
Exercise

1. Find the angle $\phi$ for the curve:
   
   (i) $\frac{2a}{r} = 1 - \cos \theta$
   
   (ii) $r^m = a^m \cos m\theta$

2. Find the angle of intersection of the curves $r = \sin \theta + \cos \delta$ and $r = 2 \sin \theta$.

3. Show that in the equiangular spiral $r = a e^{\theta \cot \alpha}$ the tangent is inclined at a constant angle the radius vector.

   [Hint: Prove that $\phi = \alpha$.] Also show that polar subtangent is $x$ and $\alpha$ and polar subnormal is $r \cot \alpha$.

4. Show that in the curve $r = a \theta$, the polar subnormal is constant and in the curve $r\theta = a$, the polar subtangent is constant.

5. For the cardioid $r = a (1 - \cos \theta)$, Prove that

   (i) Polar subtangent $= 2a\sin^2\frac{\theta}{2} \tan\frac{\theta}{2}$

   (ii) Polar tangent $= 2a\sin^2\frac{\theta}{2} \sec\frac{\theta}{2}$

   (iii) Polar normal $= 2a\sin\frac{\theta}{2}$

   (iv) Polar subnormal $= a \sin\theta$.

6. Show that logarithmic spiral $r = a e^{\theta}$ has the lengths of its polar tangent, polar normal, polar subtangent and polar subnormal each proportional to $r$.

7. Prove that the two curves $r = \frac{a}{1 - \cos \theta}$ and $r = \frac{b}{1 + \cos \theta}$ cut orthogonally.
Exercise

1. Obtain the pedal equation of the following curves:
   
   (i) \( r = a \left(1 - \cos \theta \right) \)
   
   (ii) \( \frac{2a}{r} = 1 - \cos \theta \)
   
   (iii) \( r^n = a^n \sin n\theta \)
   
   (iv) \( r = ae^{\theta \cos \alpha} \)
   
   (v) \( r^2 \cos 2\theta = a^2 \)

2. For the parabola \( \frac{2a}{r} = 1 - \cos \theta \), show that polar subtangent is \( 2a \cosec \theta \) and \( p = a \cosec \frac{\theta}{2} \).
Exercise

1. Find \( \frac{ds}{dx}, \frac{ds}{dy} \) for the following curves:

   (i) \( yc = \cos h \frac{x}{c} \)

   (ii) \( x^3 = ay^2 \).

2. Find \( \frac{ds}{d\theta} \) for the following curves

   (i) \( x = a \cos \theta, y = b \sin \theta \)

   (ii) \( x = a (\theta - \sin \theta), y = a (1 - \cos \theta) \)

   (iii) \( r^2 = a^2 \cos 2\theta \)

3. Show that for any curve

\[
\frac{ds}{d\theta} = \frac{r^2}{p}.
\]
Exercise

1. Find the radius of curvature at any point \((x, y)\) on the following curves:
   
   \(y^2 = 4ax\) 
   
   \(ay^2 = x^3\) 
   
   \(xy = c^2\)

2. Show that the radius of curvature at the point \((\cos^3 a, \sin^3 a)\) on the curve \(x^{2/3} + y^{2/3} = a^{2/3}\) is \(3a \sin \theta \cos \theta\).

3. Find the radius of curvature at the point specified on the following curves:
   
   \(\sqrt{x} + \sqrt{y} = 1\), at the point \(\left(\frac{1}{4}, \frac{1}{4}\right)\)

   \(y = 4 \sin x - \sin 2x\), at the point \(x = \frac{\pi}{2}\).
Exercise

1. Find the radius of curvature at any point on the following curves:
   
   (i) \( p^2 = ar \) (parabola)
   
   (ii) \( pr = a^2 \) (hyperbola)
   
   (iii) \( r^3 = 2ap^2 \) (cardiod)
   
   (iv) \( r^3 = a^2p \) (lemniscate)

2. In the curve \( p = \frac{r^{n+1}}{a^n} \) show that the radius of curvature varies inversely as the \((n-1)\)th power of the radius vector.

3. Prove that for the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), \( p = \frac{a^2b^2}{p^3} \) where \( p \) is the perpendicular from the centre upon the tangent at \((x, y)\).
Exercise

1. Find the radius of curvature at any point \((r, \theta)\) on the following curves:
   
   \((i)\) \(r = \frac{a}{\theta}\)

   \((ii)\) \(r = a \cos \theta\)

   \((iii)\) \(r^2 = a^2 \cos 2\theta\)

2. Show that the radius of curvature at the point on the cardioid \(r = a (1 - \cos \theta)\) is \(\frac{2}{3} \sqrt{2ar}\).

3. Establish the formula
   
   \[\rho = \frac{(u^2 + u'^2)^{3/2}}{u'(u + u'')}, \text{ where } u = \frac{1}{r}.\]

4. Prove that for any curve
   
   \[\frac{r}{\rho} = \sin \phi \left(1 + \frac{d\phi}{d\theta}\right).\]
Exercise

1. Find the asymptotes of the following curves:
   
   \((i)\) \(y^3 + 3y^2x - x^2 y - 3x^3 + y^2 - 2xy + 3x^2 + 4y + 5 = 0\)

   \((ii)\) \(y^3 + x^2y + 2xy^2 - y + 1 = 0\)

   \((iii)\) \(y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 = 1\)

   \((iv)\) \((x + y)(x + 2y + 2) = x + 9y + 2\)

   \((v)\) \(2x^3 - x^2 y + 2xy^2 + y^3 - 4x^2 + 8xy - 4x + 1 = 0\)
Exercise

Find the asymptotes of the following curves:

1. \[ y^2 \left( x^2 - a^2 \right) = x \]
2. \[ x^2 y^2 = a^2 (x^2 + y^2) \]
3. \[ r \sin \theta = 2 \cos \theta \]
4. \[ r = a \cosec \theta + b \]
5. \[ r \sin n\theta = a \]
Exercise

1. Determine the position and nature of the double points on the curve
   \[ x^3 - y^3 - 7x^2 + 4y + 15x - 13 = 0 \]

2. Determine the position and nature of the double points on the curve
   \[ x^4 - 2y^3 - 3y^2 + 2x^2 + 1 = 3 \]

3. Determine the position and nature of the double points on the curve
   \[ x^3 - 2ay^3 - 3a^2 y^2 - 2a^2 x^2 + a^4 = 0 \]

4. Determine the position and nature of the double points on the curve
   \[ x^3 + 2x^3 + 2xy - y^3 + 5x - 2y \]

5. Find the tangents at the origin to the following curves
   (i) \[ a^2 (x^2 - y^2) = x^2 y^2 \]
   (ii) \[ (x^2 + y^2) = 4a^2 xy \]
Exercise

Trace the following curves:

1. \( y^2 (2a - x) = x^2 \)
   
   [Hint: This curve is symmetrical about the \( x \)-axis. The curve does not exist for negative values of \( x \). Origin is a cusp on the curve, the \( x \)-axis being the tangent at the cusp. There is the asymptote \( 2a - x = 0 \). The curve does not exist on the right-hand side of the line \( x = 2a \).]

2. \( xy^2 = 4a^2 (2a - x) \)
   
   [Hint: Curve is symmetrical about the \( x \)-axis, it does not pass through the origin. Its asymptote is \( x = 0 \). The curve does not exist for negative values of \( x \) or positive values of \( x \) greater than \( 2a \).]

The curve cuts the \( x \)-axis at \((2a, 0)\) and the tangent to the curve at \((2a, 0)\) is parallel to the \( y \)-axis.]

There are two points of inflexion. Find these points.
3. \( ay^2 = x^2 (a - x) \)
4. \( 9ay^2 = x (x - 3a)^2 \)
5. \( x^3 + y^3 - 3axy = 0 \)
   [Hint: The curve is symmetrical about the line \( y = x \). The line \( y = x \) meets the curve at \((0, 0)\) and also at \( \left( \frac{3a}{2}, \frac{3a}{2} \right) \)]

![Diagram 1]

The curve passes through the origin and \( x = 0, y = 0 \) are the tangents, there so that origin is a node on the curve.

\( x + y + a = 0 \) is the only asymptote.

6. \( y^2 (a^2 + x^2) = x^2 (a^2 - x^2) \)

![Diagram 2]

[Hint: It is symmetrical about both the axis. It passes through the origin and \( y = \pm x \) are the two tangents there. So that the origin is a node on the curve.
It has no asymptote, \( y \) is real only when \( x \) lies between \(-a\) and \( a \).]
7. \( y^2 (a + x) = x^2 (3a - x) \).

[**Hint:** The curve is symmetrical about the \( x \)-axis and lies between \( x = -a \) and \( x = 3a \). \( x = -a \) is the asymptote.

Origin lies on the curve and \( y = \pm \sqrt{3}x \) are the two tangents at the origin. \( \therefore \) origin is a node on the curve.]

8. \( y^2 (a^2 - x^2) = x^4 \).

[**Hint:** Curve is symmetrical about both the axis.

\( x = \pm a \) are the asymptotes. No value of \( x \) can numerically exceed \( a \). The \( x \)-axis is tangent at the origin.]
B.A. (Programme) I Year

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Exercise

1. Trace the following curves:
   
   (i) \( r = a (2 \cos \theta + \cos 3\theta) \)
   
   (ii) \( r = a \cos 3\theta \)
   
   (iii) \( r = a \sin 2\theta \)

   [Hint: We will get four loops corresponding to the variation of \( \theta \) in the intervals
   \[ \left[ 0, \frac{\pi}{2} \right], \left[ \frac{\pi}{2}, \pi \right], \left[ \frac{3\pi}{2}, \frac{3\pi}{2} \right], \left[ \frac{3\pi}{2}, 2\pi \right] \] respectively.]