Lesson 1

Monotone Functions and Extreme Values

1.1 Monotone functions

Definition 1.1.1: A function $f$ defined on an interval $[a, b]$ is said to be *monotonically increasing or simply increasing* if for all $x_1, x_2$ in $[a, b]$

- $f(x_1) \leq f(x_2)$ whenever $x_1 \leq x_2$
- or $f(x_1) \geq f(x_2)$ whenever $x_1 \geq x_2$

Definition 1.1.2: A function $f$ defined on an interval $[a, b]$ is said to be *strictly increasing* (fig.1.1) if for all $x_1, x_2$ in $[a, b]$

- $f(x_1) < f(x_2)$ whenever $x_1 < x_2$
- or $f(x_1) > f(x_2)$ whenever $x_1 > x_2$

Definition 1.1.3: A function $f$ defined on an interval $[a, b]$ is said to be *monotonically decreasing or simply decreasing* if for all $x_1, x_2$ in $[a, b]$

- $f(x_1) \leq f(x_2)$ whenever $x_1 \geq x_2$
- or $f(x_1) \geq f(x_2)$ whenever $x_1 \leq x_2$

Definition 1.1.4: A function $f$ defined on an interval $[a, b]$ is said to be *strictly decreasing* (fig.1.2) if for all $x_1, x_2$ in $[a, b]$

- $f(x_1) < f(x_2)$ whenever $x_1 > x_2$
- or $f(x_1) > f(x_2)$ whenever $x_1 < x_2$

Definition 1.1.5: A function $f$ defined on an interval $[a, b]$ is said to be a *monotone (or strictly monotone) function* if $f$ is either an increasing (or strictly increasing) function or a decreasing (or strictly decreasing) function.
Criterion for Monotone functions

**Theorem 1.2.1:** If a function $f$ defined on $[a, b]$ be such that

(i) $f$ is continuous on $[a, b]$

(ii) $f'(x) > 0 \ \forall x \in [a, b]$

then $f$ is strictly increasing in $[a, b]$.

**Proof:** Let $x_1$ and $x_2$ be two points in $[a, b]$ such that $x_1 < x_2$

We know that

$$f'(x_1) = \lim_{x \to x_1} \frac{f(x) - f(x_1)}{x - x_1}$$

Choose $x = x_2$, then

$$f'(x_1) = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \text{.................................(I)}$$

Now $x_1 < x_2 \Rightarrow (x_2 - x_1) > 0$ and by hypothesis (ii), $f'(x_1) > 0$

$\therefore (I) \Rightarrow f(x_2) - f(x_1) > 0$

i.e. $f(x_2) > f(x_1)$
Hence, in view of definition 1.1.2, f is strictly increasing.

**Theorem 1.2.2:** If a function f defined on [a, b] be such that

(i) \( f \) is continuous on \([a, b]\)

(ii) \( f'(x) < 0 \ \forall x \in [a, b] \)

then \( f \) is strictly decreasing in \([a, b]\).

**Proof:** Let \( x_1 \) and \( x_2 \) be two points in \([a, b]\) such that \( x_1 < x_2 \).

Proceeding as in theorem 1.2.1, due to (ii) of the hypothesis, the relation (I) gives,

\[
 f(x_2) - f(x_1) < 0
\]

i.e. \( f(x_1) > f(x_2) \)

Hence, in view of definition 1.1.4, \( f \) is strictly decreasing.

**Example 1:** Show that the function \( 3x^3 - 9x^2 + 9x + 7 \) is strictly increasing in every interval.

**Solution:** Let \( f(x) = 3x^3 - 9x^2 + 9x + 7 \).

then \( f'(x) = 9x^2 - 18x + 9 \)

Since \( f(x) \), being a polynomial in \( x \) is continuous for all \( x \), therefore for \( f(x) \) to be strictly increasing (by theorem 1.2.1) \( f'(x) > 0 \)

\[
\Rightarrow 9x^2 - 18x + 9 > 0
\]

\[
\Rightarrow 9(x^2 - 2x + 1) > 0
\]

\[
\Rightarrow 9(x^2 - 2x + 1) > 0
\]

\[
\Rightarrow (x - 1)^2 > 0, \text{ which is always true, being a perfect square.}
\]

\[
\Rightarrow \text{ For all real values of } x \neq 1, \text{ the function } f(x) \text{ is strictly increasing.}
\]

For \( x = 1, f'(x) = 0 \)

i.e. \( f'(x) \geq 0 \)

\[
\Rightarrow \text{the function } f(x) \text{ is increasing but not strictly increasing.}
\]

**Example 2:** Separate the intervals in which the function
\[ f(x) = x^3 - 6x^2 + 9x + 1 \]
is increasing and decreasing.

**Solution:** We have \( f'(x) = 3x^2 - 12x + 9 \)

As \( f \) is continuous for all \( x \), therefore, by theorem 1.2.1, the function increases if \( f'(x) > 0 \) for all \( x \).

\[
\text{i.e. if } 3x^2 - 12x + 9 > 0 \\
\Rightarrow x^2 - 4x + 3 > 0 \text{ i.e. } (x - 1)(x - 3) > 0 \\
\Rightarrow \text{either } (x - 1) > 0 \text{ and } (x - 3) > 0 \\
\quad \text{or } (x - 1) < 0 \text{ and } (x - 3) < 0 \\
\Rightarrow \text{either } x > 1 \text{ and } x > 3 \text{ or } x < 1 \text{ and } x < 3 \\
\Rightarrow \text{either } x > 3 \text{ or } x < 1 \Rightarrow x \in [3, \infty \cup -\infty, 1] \\
\]

Thus function increases in \([3, \infty \cup -\infty, 1]\).

For a function to be decreasing, by theorem 1.2.2,

\[ f'(x) = 3x^2 - 12x + 9 < 0 \]

Proceeding as above, we have

\[
\text{either } (x - 1) > 0 \text{ and } (x - 3) < 0 \\
\text{or } (x - 1) < 0 \text{ and } (x - 3) > 0 \\
\text{i.e. } f \text{ decreases in } [1,3].
\]

**Example 3:** A sign chart for the first derivative of a function \( f \) is presented. Assuming that \( f \) is continuous everywhere, find the intervals on which \( f \) is increasing or decreasing:

<table>
<thead>
<tr>
<th>Interval</th>
<th>sign of ( f'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x &lt; 1 )</td>
<td>-</td>
</tr>
<tr>
<td>( 1 &lt; x &lt; 2 )</td>
<td>+</td>
</tr>
<tr>
<td>( 2 &lt; x &lt; 3 )</td>
<td>+</td>
</tr>
<tr>
<td>( 3 &lt; x &lt; 4 )</td>
<td>-</td>
</tr>
<tr>
<td>( 4 &lt; x )</td>
<td>-</td>
</tr>
</tbody>
</table>
Solution: Since the table shows that for \( x < 1, 3 < x < 4 \) and \( 4 < x \), the sign of the first derivative is negative i.e. \( f'(x) < 0 \), so, in view of the theorem 1.2.2, the function is decreasing in these intervals i.e. \( f \) is decreasing for \( x \in ]-\infty,1[ \cup 3, 4[ \cup 4, \infty[ \) i.e. \( f \) decreases in \( ]-\infty,1[ \cup 3, \infty[ \). Similarly, as \( f'(x) > 0 \) for \( 1 < x < 2 \) and \( 2 < x < 3 \), so, in view of the theorem 1.2.1, the function \( f \) increases in \( ]1, 2[ \cup 2, 3[ \) i.e. in \( ]1,3[ \).

1.2.3 Applications of monotone functions
We can establish inequalities using the concept of monotone functions. Let us prove a few in the following examples.

Example 4: Show that, for all \( x > 0 \)
\[
e^x > 1 + x
\]
Solution: Define a function \( f(x) \) as
\[
f(x) = e^x - (1 + x)
\]
As polynomial functions are continuous and exponential function is also continuous for all \( x > 0 \), therefore \( f \) is continuous for all \( x > 0 \).
Now \( f'(x) = e^x - 1 \) for all \( x > 0 \)
Define another function \( g(x) = f'(x) = e^x - 1 \) for all \( x > 0 \)
Now \( g \) is continuous and \( g'(x) = e^x \) for all \( x > 0 \)
We know that \( e^x > 0 \) for all \( x > 0 \) \( \Rightarrow g'(x) > 0 \) for all \( x > 0 \)
\[\Rightarrow g \text{ is a strictly increasing function for all } x > 0\]
\[\therefore x > 0 \Rightarrow g(x) > g(0) \text{ (by definition 1.1.2)}\]
i.e. \( e^x - 1 > e^0 - 1 \Rightarrow e^x - 1 > 0 \ \forall x > 0 \)
\[\Rightarrow f'(x) > 0 \ \forall x > 0\]
\[\Rightarrow f \text{ is an increasing function of } x\]
\[\therefore x > 0 \Rightarrow f(x) > f(0)\]
i.e. \( e^x - (1 + x) > e^0 - (1 + 0) = 0\)
\[\Rightarrow e^x > (1 + x) \ \forall x > 0\]
Example 5: Show that, for all $x > 0$

$$x - \frac{x^2}{2} \leq \log (1+x) \leq x - \frac{x^2}{2(1+x)}$$

Solution: Consider two functions $f(x)$ and $g(x)$, defined as follows:

$$f(x) = \log (1+x) - x - \frac{x^2}{2(1+x)}$$

and

$$g(x) = x - \frac{x^2}{2(1+x)} - \log (1 + x)$$

Now $f'(x) = \frac{1}{1+x} - 1 + x = \frac{x^2}{1+x} > 0 \forall x > 0$

i.e. $f'(x) > 0 \forall x > 0$

$\Rightarrow f$ is an increasing function $\forall x > 0$

$\therefore x > 0 \Rightarrow f(x) > f(0)$

$\Rightarrow \log (1+x) - x + \frac{x^2}{2} > \log (1+0) = 0$

$\Rightarrow \log (1+x) > x - \frac{x^2}{2} \quad \text{..................................................(I)}$

Now $g'(x) = \frac{x^2}{2(1+x)^2} > 0 \forall x > 0$

$\Rightarrow g$ is an increasing function $\forall x > 0$

$\Rightarrow x > 0 \Rightarrow g(x) > g(0)$

$\Rightarrow < x - \frac{x^2}{2(1+x)} - \log (1 + x) > - \log(1+0)$

$\Rightarrow x - \frac{x^2}{2(1+x)} > \log (1 + x) \quad \text{..................................................(II)}$

Combining (I) and (II) we get the required inequality.
**Exercise 1.1**

1. Separate the intervals in which the following functions are increasing or decreasing:
   
   (i) \( f(x) = 2x^3 - 15x^2 + 36x + 1 \)
   (ii) \( f(x) = x^3 + 6x^2 + 12x - 7 \)
   (iii) \( f(x) = 2x^3 - 9x^2 + 12x - 5 \)
   (iv) \( f(x) = (x^4 + 6x^3 + 17x^2 + 32x + 32)e^{-x} \)

2. A sign chart for the first derivative of a function \( f \) is presented. Assuming that \( f \) is continuous everywhere, find the intervals on which \( f \) is increasing or decreasing:

   (a) Interval | sign of \( f'(x) \)
   1. \( x < 1 \) | +
   2. \( 1 < x < 3 \) | +
   3. \( 3 < x \) | +

   (b) Interval | sign of \( f'(x) \)
   1. \( x < -2 \) | --
   2. \( -2 < x < 0 \) | +
   3. \( 0 < x < 1 \) | --
   4. \( 1 < x \) | +

3. Establish the following inequalities:
   
   (i) \( e^x > 1 + x + \frac{1}{2}x^2 \) \( \forall \ x > 0 \)
   (ii) \( e^x > 1 - x \) \( \forall \ x > 0 \)
   (iii) \( x > \sin x > x - \frac{1}{6}x^3 \) \( \forall \ x > 0 \)
   (iv) \( \tan x > x \), if \( 0 < x < \frac{\pi}{2} \)
   (v) \( (1 + x)^{1/3} < 1 + \frac{1}{3}x \)

4. Show that
   
   (i) \( x (\sin x)^{-1} \) increases for \( 0 < x < \frac{\pi}{2} \)
   (ii) \( x (\tan x)^{-1} \) decreases for \( 0 < x < \frac{\pi}{2} \)
1.3 Extreme Values
Let $f$ be a function defined on an open interval $[a, b]$ and $c \in ]a, b[$. Then:

**Definition 1.3.1:** $f$ is said to have *relative maximum value* (or simply a maximum) at the point $c$ if there exist some $\delta > 0$ such that

- $f(x) < f(c)$ for all $x \in ]c - \delta, c + \delta[ \setminus x \neq c$
- i.e. $f(x) < f(c)$ for all $0 < |x - c| < \delta$

In that case we say $f(c)$ is the *relative maximum value* at $c$.

**Definition 1.3.2:** $f$ is said to have *relative minimum value* (or simply a minimum) at the point $c$ if there exist some $\delta > 0$ such that

- $f(x) > f(c)$ for all $x \in ]c - \delta, c + \delta[ \setminus x \neq c$
- i.e. $f(x) > f(c)$ for all $0 < |x - c| < \delta$

In that case we say $f(c)$ is the *relative minimum value* at $c$.

**Definition 1.3.3:** $f$ is said to have *relative extremum* (or simply an extremum) at the point $c$ if $f$ has either a relative maximum or a relative minimum at $c$.

**Definition 1.3.4:** The points $x = c$ for which the function $f(x)$ has relative extremum are called *critical points or stationary points*.

**Remark 1:** Though, the terms -- relative extremum and extremum of a function are very different from the point of view of a mathematician. But at the beginner’s level, both terms can be used with the same meaning.

**Theorem 1.3.5:** A necessary condition for the existence of an extreme value

**Statement:** Let $f$ be a function defined on an interval $[a, b]$ and let $f$ be derivable at a point $c \in ]a, b[$. If $f$ has an extreme value at $c$ then $f'(c) = 0$.

**Proof:** We shall prove the result *geometrically:*
In the figure 1.3, the function \( f \) has maximum value at the point \( x = c \) in the interval \( c-\delta, c+\delta \). Also, in this interval, the value of the function increases from \( x = c-\delta \) till \( x = c \) and then starts decreasing till \( x = c+\delta \). At the point \( x = c \) the tangent drawn is exactly parallel to \( x \)-axis i.e. at the point of maximum the function is changing its behavior from increasing to decreasing. So, in view of the definitions 1.1.1 to 1.1.4, at the point of maximum, the sign of derivative of the function changes from positive to negative. But \( f \), being differentiable at \( c \), is continuous at \( x = c \). Therefore, at some point in \( c-\delta, c+\delta \), the derivative of the function should be zero i.e. at some point in \( c-\delta, c+\delta \), the tangent drawn is parallel to \( x \)-axis (see the tangent drawn at \( x = c \) in the figure). Hence, the necessary condition for the existence of maximum at \( x = c \) is that \( f'(c) = 0 \).

A similar argument holds for minimum. At the point of minimum i.e. at \( x = d \), as is obvious from the figure 1.3, the function changes its behavior from decreasing to increasing and hence again at the point of minimum the derivative is zero.

![Diagram of function showing points of maximum and minimum](image-url)
**Theorem 1.3.6:** A sufficient condition for the existence of an extreme value (Second derivative test):

**Statement:** Let \( f \) be a function defined on an interval \([a, b]\) and let \( f \) be twice differentiable at a point \( c \in ]a, b[\). Then, if

(i) \( f'(c) = 0 \) and \( f''(c) < 0 \) then \( f \) has a maximum value at \( x = c \).

(ii) \( f'(c) = 0 \) and \( f''(c) > 0 \) then \( f \) has a minimum value at \( x = c \).

**Proof:** Not needed at this point.

**Remark 2:** We may have the functions for which second derivative also vanishes at the critical points i.e. the points at which the first derivative is zero. In other words, \( f'(c) = 0 \) and \( f''(c) = 0 \). In this case, if \( f'''(c) \neq 0 \), then the point \( x = c \) is called a **point of inflection**. In fact, the points of inflection are the points of relative extremum for the first order derivative \( f'(x) \). Precisely:

**Definition 1.3.7:** For a differentiable function \( f(x) \), a **point of inflection** is a point at which the first order derivative changes its behavior from an increasing function to a decreasing function or from a decreasing function to an increasing function.

Since at inflection points, the second derivative test fails. So, to discuss the existence of extremum at such points we have to use the following result:

**Theorem 1.3.8:** Let \( f \) be a function defined on an interval \([a, b]\) and let \( f \) be differentiable at a point \( c \in ]a, b[\). Let \( f''(c) = f'''(c) = \ldots = f^{(n-1)}(c) = 0 \) and \( f^{(n)}(c) \neq 0 \) then

(i) if \( n \) is odd, \( f \) has neither maximum nor minimum at \( x = c \).

(ii) if \( n \) is even and \( f^{(n)}(c) < 0 \), \( f \) has maximum at \( x = c \).

(iii) if \( n \) is even and \( f^{(n)}(c) > 0 \), \( f \) has minimum at \( x = c \).

**Example 6:** Investigate the following function for extreme values.

\[ f(x) = x^4 - 2x^3 - 2x^2 + 6x - 5 \]
Solution: We have

\[ f'(x) = 4x^3 - 6x^2 - 4x + 6 \]
\[ = 2x^2 (2x - 3) - 2 (2x - 3) \]
\[ = (2x - 3) (2x^2 - 2) \]

Therefore \( f'(x) = 0 \Rightarrow x = 3/2, +1, -1 \)

Now \( f''(x) = 12x^2 - 12x - 4 \) which implies \( f''(3/2) = 27 - 18 - 4 = 5 > 0 \)

i.e. \( f \) has minimum at \( x = 3/2 \)

Also \( f''(+1) = -4 < 0 \) i.e. \( f \) has maximum at \( x = 1 \)

Further \( f''(-1) = 20 > 0 \) i.e. \( f \) has minimum at \( x = -1 \)

Example 7: Show that \( f \) has neither maximum nor minimum at \( x = 0 \) if the function \( f \) is defined as \( f(x) = x^5 - 5x^4 + 5x^3 - 1 \)

Solution: We have

\[ f'(x) = 5x^4 - 20x^3 + 15x^2 \]
\[ = 5x^2 (x - 1) (x - 3) \]

Therefore \( f'(x) = 0 \Rightarrow x = 0, +1, 3 \)

Now \( f''(x) = 20x^3 - 60x^2 - 30x \) which implies \( f''(0) = 0 \)

But \( f'''(x) = 60x^2 - 120 \Rightarrow f''''(0) \neq 0 \)

Refer to Theorem 1.3.8 (i), since \( n = 3 \) is odd, so \( f \) has neither maximum nor minimum at \( x = 0 \).

Example 8: Find the points of maximum and minimum, if any, for the function \( f(x) = x + \sin 2x \) in the interval \([0, 2\pi]\). Also find the maximum value and the minimum value of the function in this interval.

Solution: Since \( f'(x) = 1 + 2 \cos 2x \). Therefore \( f'(x) = 0 \Rightarrow \cos 2x = -\frac{1}{2} \).

In the interval \([0, 2\pi]\), this equation gives the solution as \( 2x = 2\pi/3 \) or \( 4\pi/3 \)

i.e. \( x = \pi/3 \) or \( x = 2\pi/3 \)

Now \( f''(x) = -4 \sin 2x \)
So, \( f''(\pi/3) = -4 \sin(2\pi/3) = -2\sqrt{3} \) and \( f''(2\pi/3) = -4 \sin(4\pi/3) = 2\sqrt{3} \)

i.e. \( f''(\pi/3) < 0 \) while \( f''(2\pi/3) > 0 \)

Hence \( f \) has maximum at \( x = \pi/3 \) and minimum at \( x = 2\pi/3 \)

The maximum value of the function is \( f(\pi/3) = \pi/3 + \sin(2\pi/3) = \pi/3 + \sqrt{3}/2 \)

The minimum value of the function is \( f(2\pi/3) = 2\pi/3 + \sin(4\pi/3) = \pi/3 - \sqrt{3}/2 \).

**Example 9:** Show that the function \( f \), defined by

\[
f(x) = \begin{cases} 
(-1)^m x^m (1-x)^n & \text{if } x < 0 \\
0 & \text{if } x = 0 \\
x^m (1-x)^n & \text{if } 0 < x < 1 \\
0 & \text{if } x = 1 \\
x^m (x-1)^n & \text{if } x > 1 
\end{cases}
\]

has a maximum value \( m^m n^n / (m + n)^{m+n} \), \( m \) and \( n \) being positive.

**Solution:** Using the definition of the mode function we can rewrite the function as

\[
f(x) = \begin{cases} 
(-1)^m x^m (1-x)^n & \text{if } x < 0 \\
0 & \text{if } x = 0 \\
x^m (1-x)^n & \text{if } 0 < x < 1 \\
0 & \text{if } x = 1 \\
x^m (x-1)^n & \text{if } x > 1 
\end{cases}
\]

so that

\[
f'(x) = \begin{cases} 
(-1)^m x^{m-1} (1-x)^{n-1} (m - (m+n)x) & \text{if } x < 0 \\
0 & \text{if } x = 0 \\
x^{m-1} (1-x)^{n-1} (m - (m+n)x) & \text{if } 0 < x < 1 \\
0 & \text{if } x = 1 \\
x^m (x-1)^n ((m+n)x - m) & \text{if } x > 1 
\end{cases}
\]

For extreme values \( f'(x) = 0 \Rightarrow x = \frac{m}{(m+n)} \), \( 0, 1 \)

Verify that \( f''(x) < 0 \) for \( x = m/ (m+n) \) and \( f''(x) = 0 \) for \( x = 0 \) and \( x = 1 \)

Thus, the given function has maximum at the point \( x = m/ (m+n) \).
The maximum value is

\[
f\left(\frac{m}{m+n}\right) = \frac{m^m}{(m+n)^m} \left(1 - \frac{m}{m+n}\right)^n = \frac{m^m n^n}{(m+n)^{m+n}}
\]

**Exercise 1.2**

1. Find the extreme values for the following functions
   
   (i) \( f(x) = x^4 + 4x^3 - 2x^2 - 12x + 7 \)
   
   (ii) \( f(x) = (x - 1)(x - 2)(x - 3) \)
   
   (iii) \( f(x) = 12x^5 - 45x^4 + 40x^3 + 6 \)
   
   (iv) \( f(x) = x^3 + 3px + q \)
   
   (v) \( f(x) = 3 \left| x \right| + 4 \left| x - 1 \right| \) for all real \( x \)
   
   (vi) \( f(x) = \sin x (1 + \cos x) \) for \( x \in [0, 2\pi] \)
   
   (vii) \( f(x) = x^2 - x + 2 \)
   
   (viii) \( f(x) = 2x^3 + 3x^2 - 36x + 10 \)
   
   (ix) \( f(x) = x^2 (x - 1)^3 \)
   
   (x) \( f(x) = 4x^{-1} - (x - 1)^{-1} \), \( x \in \mathbb{R} \sim \{0,1\} \)

2. Divide 100 into two parts such that the sum of their squares is minimum.

3. Show that \( y = x + x^{-1} \) has two critical points, one gives maximum and other gives minimum but the functional value of the latter point is larger than that of former.

4. Show that \( f(x) = (x + 1)^2 / (x + 3)^3 \) has a maximum value \( 2/27 \) and a minimum value \( 0 \).

5. Find the values of \( a, b, c \) and \( d \) so that the function

   \[
f(x) = ax^3 + bx^2 + cx + d
\]

   has a relative minimum at \((0,0)\) and a relative maximum at \((1,1)\).
MEAN VALUE THEOREMS

2.1 Introduction

In earlier sections, we have already learnt about the limit and continuity of the real valued functions at a point or in an interval. Let us recall these definitions for the ready reference:

Let \( f: [a, b] \rightarrow [a, b] \) be a real valued function and \( x_0 \in ]a, b[ \) be any arbitrary point. Then:

**Definition 2.1.1:** \( f \) is said to be **continuous at the point** \( x_0 \) if

\[
R.H.L. = f(x_0 + h) = f(x_0 - h) = L.H.L.
\]

**Definition 2.1.2:** \( f \) is said to be **differentiable or derivable at** \( x_0 \) if

\[
R.H.D. = \frac{f(x_0 + h) - f(x_0)}{h} = \frac{f(x_0 - h) - f(x_0)}{-h} = L.H.D.
\]

**Definition 2.1.3:** \( f \) is said to be **continuous in an interval** \([a, b]\) if \( f \) is continuous at each point of the interval (see 2.1.1) Similarly \( f \) is said to be **differentiable in an interval** \([a, b]\) if \( f \) is differentiable at each point of the interval (see 2.1.2).

2.1.4: Following is a very special property of continuous functions to be stated without proof:

*If a function is continuous in a closed interval \([a, b]\), then \( f \) is bounded and attains its bounds which then become the maximum and minimum values of the function in \([a, b]\).*

In another words if \( f \) is continuous in the closed interval \([a, b]\), then there exist two points \( c \) and \( d \) in the interval \([a, b]\) such that \( f(c) \leq f(x) \leq f(d) \). Obviously, in that case \( f(c) \) and \( f(d) \) are, respectively, the smallest and the greatest values of the function \( f \) in the interval \([a, b]\).
2.2 Rolle’s Theorem

2.2.1 Statement: Let \( f \) be a function defined on \([a, b]\) such that

(i) \( f \) is continuous in \([a, b]\)

(ii) \( f \) is differentiable in \(]a, b[\)

(iii) \( f(a) = f(b) \)

Then there exist a point \( c \) in \(]a, b[\) such that \( f'(c) = 0 \).

Proof: As \( f \) is a real valued function, two cases arise:

Case (I): \( f \) is a constant function i.e. \( f(x) = k \) for all \( x \in [a, b] \), (\( k \) is a constant). Then \( f'(x) = 0 \) for all \( x \in [a, b] \) and hence the theorem is true in this case.

Case (II): \( f \) is not a constant function.

Since \( f \) is continuous in \([a, b]\) therefore, by 2.1.4, there exist \( x_1 \) and \( x_2 \) in \([a, b]\) such that \( f(x_1) \leq f(x) \leq f(x_2) \) where \( f(x_1) = \text{min.}\{f(x) : x \in [a, b]\} \) and \( f(x_2) = \text{Max.}\{f(x) : x \in [a, b]\} \)

If \( f(a) = f(x_1) \) then, in view of (iii) of the hypothesis, \( f(b) = f(x_2) = f(a) = f(x_1) \) i.e. \( f \) is a constant function, a contradiction to case II. Hence \( f(a) \neq f(x_1) \). Similarly \( f(b) \neq f(x_2) \). Thus at least one of \( x_1 \) and \( x_2 \) must be in the open interval \( ]a, b[\). Also these points being points of maximum and minimum must have first derivative as zero. Thus there exist a point \( c \) in the open interval \( ]a, b[\) such that \( f'(c) = 0 \).

Remark 1: Rolle’s theorem ensures us about the existence of at least one real number \( c \) in \( ]a, b[\) with \( f'(c) = 0 \). We may have more than one such numbers (see fig.2.1).

Example 1: Verify Rolle’s theorem for the function

\[ f(x) = x^4 - 3x^2 + 4 \quad \text{in} \quad [-4, 4] \]

Solution: \( f(x) \), being a polynomial function is continuous in \([-4, 4]\) and derivable in \([-4,4] \). Also \( f(-4) = f(4) = 212 \).

i.e. all conditions of the hypothesis of Rolle’s theorem are satisfied.

Now \( f'(x) = 0 \) implies \( 4x^3 - 6x = 0 \) which gives \( x = 0, \pm \sqrt[3]{3/2} \). As all these three numbers
belong to the interval ]−4,4[. Therefore, the conclusion i.e. there exists at least one
c ∈ ]−4,4[ such that f ′(c) = 0, is also verified.

**Remark 2:** The conclusion of Rolle’s theorem may not be true for a function which does not satisfy even one of the conditions of the hypothesis:

**Example 2:** For the function f(x) = |x| in [−1,1], the theorem does not hold as the function is not derivable at x = 0 (verify as done in section 3).

**Example 3:** Consider the function f(x) = \( \begin{cases} x & \text{for } 0 < x \leq 1 \\ 1 & \text{for } x = 0 \end{cases} \)
Since at x = 0 the function is not continuous therefore Rolle’s theorem does not hold for f (verify as done in section 3).

**Example 4.** Let f(x) = x for all x in [1,2] Here the function f satisfies the condition (i) and (ii) of the Rolle’s theorem but since \( f(1) \neq f(2) \) the theorem is still not valid.

**Remark 3:** It may be possible that all conditions of the hypothesis are satisfied and there exist more than one real numbers c with f ′(c) = 0 but all of these points are not in the open interval ] a, b[ e.g.

**Example 5.** Consider \( f(x) = x (x + 3) e^{-x/2} \) in [−3,0].
Since x (x +3), being a polynomial in x, is differentiable in ]−3,0[ and \( e^{-x/2} \) is also derivable in ]−3,0[ so f(x), being product of two derivable functions, is derivable in ]−3,0[. Obviously then, f is continuous also in ]−3,0[.
Further f (−3) = 0 = f (0).
Thus all conditions of the hypothesis of the Rolle’s theorem are verified.
Now f ′(x) = 0 gives two values of x i.e. x = −2 and x = 3 out of which 3 \( \not\in \) ]−3,0[. But
since the conclusion of the theorem needs at least one $c$ in $]a, b[,$ so in this case we say that the theorem holds.

2.2.2 Geometrical significance of Rolle’s Theorem

Interpreting each condition of the hypothesis geometrically, the Rolle’s Theorem states:

Let a curve $y = f(x)$ be drawn continuously between two points $a$ and $b$ such that the tangents can be drawn at every point between $a$ and $b$ (not necessarily at $a$ and $b$). If the ordinates $f(a)$ and $f(b)$ at the points $a$ and $b$ respectively are equal, then there always exist at least one point $c$ between $a$ and $b$, $c \neq a, c \neq b$ at which the tangent drawn to the curve is parallel to $x$-axis (see fig 2.1).

![Graph of Rolle's Theorem](attachment:image.png)

Fig. 2.1

In the figure 2.1, the points A, B and C give the required points $c \in ]a, b[$ of the Rolle’s Theorem.

**Example 6:** Show that between any two roots of the equation $e^x \cos x = 1$, there exists at least one root of the equation $e^x \sin x - 1 = 0.$
Solution: Let \( \alpha \) and \( \beta \) be the roots of the equation \( e^x \cos x = 1 \).

\[ \Rightarrow e^\alpha \cos \alpha = 1 \quad \text{and} \quad e^\beta \cos \beta = 1 \]

\[ \Rightarrow e^{-\alpha} - \cos \alpha = 0 \quad \text{and} \quad e^{-\beta} - \cos \beta = 0 \].................(1)

Define a function \( f \) as follows:

\[ f(x) = e^{-x} - \cos x \quad \text{for all} \quad x \in [\alpha, \beta] \]

As \( \cos x \) and \( e^{-x} \) are continuous in \([\alpha, \beta]\), so \( f(x) \) is continuous in \([\alpha, \beta]\).

Also \( f'(x) = -e^{-x} \sin x \quad \text{for all} \quad x \in [\alpha, \beta] \)

Further, in view of (1),

\[ f(\alpha) = e^{-\alpha} - \cos \alpha = 0 = e^{-\beta} - \cos \beta = f(\beta) \quad \text{i.e.} \quad f(\alpha) = f(\beta) \]

Hence by Rolle’s theorem \( \exists \) at least one \( c \in ]\alpha, \beta[ \) such that \( f'(c) = 0 \)

i.e. \( -e^{-c} + \sin c = 0 \Rightarrow e^c \sin c - 1 = 0 \)

i.e. \( c \) is a root of the equation \( e^x \sin x - 1 = 0 \)

i.e. \( \exists \) at least one \( x \in ]\alpha, \beta[ \) such that \( e^x \sin x - 1 = 0 \).

Thus \( \exists \) at least one root of \( e^x \sin x - 1 = 0 \) in \([\alpha, \beta]\).

Example 7: Prove that if \( a_0, a_1, a_2, \ldots, a_n \) are real numbers such that

\[ \frac{a_0}{n+1} + \frac{a_1}{n} + \ldots + \frac{a_{n-1}}{2} + a_n = 0 \].................(1)

then there exists at least one real number \( x \) between 0 and 1 such that

\[ a_0 x^n + a_1 x^{n-1} + \ldots + a_n = 0 \]

Solution: Define a function \( f \) as follows:

\[ f(x) = \frac{a_0}{n+1} x^{n+1} + \frac{a_1}{n} x^n + \ldots + \frac{a_{n-1}}{2} x^2 + a_n x = 0 \]

\( f \) being a polynomial in \( x \) is continuous in \([0,1]\) and derivable in \([0,1]\).

Now \( f(0) = 0 \) and, in view of the condition (1) of the hypothesis
\[ f(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \ldots + \frac{a_{n-1}}{2} + a_n = 0. \]

Thus by Rolle’s theorem \( \exists \) at least one \( x \in ]0,1[ \) such that \( f'(x) = 0 \)

\[ \Rightarrow f'(x) = \frac{a_0}{n+1} (n+1)x^n + \frac{a_1}{n} nx^{n-1} + \ldots + \frac{a_{n-1}}{2} 2x + a_n = 0 \]

\[ \Rightarrow a_0 x^n + a_1 x^{n-1} + \ldots + a_n = 0 \]

**Example 8:** Prove that if \( f \) is any polynomial and \( f' \) the derivative of \( f \), then between any two consecutive zeros of \( f' \), there lies at the most one zero of \( f \).

**Solution:** Let \( \alpha \) and \( \beta \) be the two consecutive zeros of \( f' \).

\[ \Rightarrow f'(\alpha) = 0 \text{ and } f'(\beta) = 0 \]

If possible, let there be two zeros of \( f \) between \( \alpha \) and \( \beta \). Let these be \( p \) and \( q \)

\[ \Rightarrow f(p) = 0 \text{ and } f(q) = 0 \]

Without loss of generality we may assume \( \alpha \leq p < q \leq \beta \)

As \( f \) is a polynomial in \( x \) so \( f \) is continuous in \( [\alpha, \beta] \) and hence in \( [p, q] \) and \( f \) is derivable in \( ]\alpha, \beta[ \) and hence in \( ]p, q[ \). Therefore, by Rolle’s theorem \( \exists \) at least one \( r \in ]p, q[ \) such that \( f'(r) = 0 \)

i.e. there exist at least one zero of the derivative polynomial \( f' \).

Since \( r \in ]p, q[ \subset ]\alpha, \beta[ \Rightarrow \alpha \leq p < r < q \leq \beta \) i.e. \( r \) is another zero of \( f' \)

between \( \alpha \) and \( \beta \), which means \( \alpha \) and \( \beta \) are not the consecutive zeros of \( f' \), which is a contradiction. Thus our assumption is wrong and hence the proof.
Exercise 2.1

1. Verify Rolle’s theorem for the following functions:
   (i) \( f(x) = x^2 \), \( x \in [-1,1] \)
   (ii) \( f(x) = x^3 - 4x \), \( x \in [-2,2] \)
   (iii) \( f(x) = \cos x \), \( x \in [-\pi/2,\pi/2] \)
   (iv) \( f(x) = (x - 4)(x - 2)(x - 3) \), \( x \in [2,4] \)
   (v) \( f(x) = (x - 5)(x - 6) e^{-x} \), \( x \in [5,6] \)

2. Examine the validity of the hypothesis and conclusion of Rolle’s theorem for the function \( f \) defined on \([a, b]\) in each of the following cases:
   (i) \( f(x) = x^{5/2} \), \( a = 1, b = 5 \)
   (ii) \( f(x) = |x - 1| \), \( a = -2, b = 2 \)
   (iii) \( f(x) = 2 + (x - 1)^{3/2} \), \( a = 0, b = 2 \)
   (iv) \( f(x) = (x + 3)^{1/2} \), \( a = -3, b = 3 \)

3. Discuss the validity of the Rolle’s theorem for \( f(x) = (x - a)^m (x - b)^n \) in \([a, b] \); \( m, n \) being positive integers.

4. By considering the function \( f(x) = (x - 2) \log x \) in \([1,2]\), show that the equation \( x \log x = 2 - x \) is satisfied by at least one value of \( x \) lying between 1 and 2.

5. Let \( f(x) = x^{2/3} \), \( a = -1, b = 8 \). Show that there is no real number \( c \) in \((a,b)\) such that
   \[ f'(c) = \frac{f(b) - f(a)}{b - a} \]

6. Use Rolle’s Theorem to show that the equation \( x^3 + 4x - 1 = 0 \) has exactly one real root.
2.3 Lagrange’s Mean Value Theorem

2.3.1 Statement: Let $f$ be a function defined on $[a, b]$ such that

(i) $f$ is continuous in $[a, b]$

(ii) $f$ is differentiable in $]a, b[$

Then there exists one point $c$ in $]a, b[$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof: Define a function $F$ on $[a, b]$ as follows:

$$F(x) = f(x) + Ax$$

where $A$ is a constant to be chosen suitably later.

(1) As $f$ is given to be continuous on $[a, b]$ and $Ax$, being a polynomial in $x$ is continuous on $[a, b]$ so $F(x)$ is continuous on $[a, b]$.

(2) Also since $f(x)$ and $Ax$ are both differentiable in $]a, b[$ therefore $F(x)$ is derivable in $]a, b[$.

(3) Choose the constant $A$ such that $F(a) = F(b)$.

(1), (2) and (3) show that $F$ satisfies the conditions of the Rolle’s theorem, therefore $\exists$ a real number $c \in ]a, b[$ such that $F'(c) = 0$

$$\Rightarrow f'(c) + A = 0$$

$$\Rightarrow f'(c) = -A \quad \text{.................................................................(I)}$$

Also, by (3), $F(a) = F(b) \Rightarrow f(a) + Aa = f(b) + Ab$

$$\Rightarrow -A = \frac{f(b) - f(a)}{b - a} \quad \text{.................................................................(II)}$$

Combining (I) and (II) we get:

$$f'(c) = -A = \frac{f(b) - f(a)}{b - a}$$
\[ i.e. \quad f'(c) = \frac{f(b) - f(a)}{b - a} \]

which is required.

**Corollary 1:** If \( f \) satisfies the conditions of Lagrange’s mean value theorem and if \( f'(x) = 0 \ \forall \ x \in ]a, b[ \), then \( f \) is constant on \([a, b]\).

**Proof:** Let \( x_1 \) and \( x_2 \) be two points in \([a, b]\) such that \( x_1 < x_2 \). Obviously \( f \) satisfies the conditions of the Lagrange’s mean value theorem in \([x_1, x_2]\). Therefore there exists one point \( c \) in \([x_1, x_2]\) such that

\[ \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) = 0 \]

\[ \Rightarrow f(x_2) - f(x_1) = 0 \quad i.e. \quad f(x_2) = f(x_1) \ \forall x_1 \text{ and } x_2 \in [a, b] \]

As \( x_1 \) and \( x_2 \) were chosen arbitrarily in \([a,b]\), we can say that for any two points of \([a,b]\) the value of \( f(x) \) is same. 

\[ \Rightarrow f \text{ is a constant function on } [a, b]. \]

**Corollary 2:** If \( f \) and \( g \) are continuous in \([a, b]\) and \( f'(x) = g'(x) \ \forall \ x \in ]a, b[ \), then \( f - g \) is constant on \([a, b]\) (or \( f \) and \( g \) differ by a constant).

**Proof:** Define a function \( H(x) \) on \([a, b]\) such that \( H(x) = f(x) - g(x) \ \forall x \in [a, b] \). Then \( H'(x) = f'(x) - g'(x) = 0 \) (by hypothesis). Therefore, by corollary 1, the function \( H \) is constant on \([a, b]\) i.e. \( f(x) - g(x) = (f - g)(x) \) is constant on \([a, b]\).

**Corollary 3:** If a function \( f \) defined on \([a, b]\) be such that

(i) \( f \) is continuous on \([a, b]\)

(ii) \( f'(x) > 0 \ \forall x \in [a, b], \)

then \( f \) is strictly increasing in \([a, b]\).
**Proof:** Let \( x_1 \) and \( x_2 \) be two points in \([a, b]\) such that \( x_1 < x_2 \).

By the given conditions of the hypothesis, \( f \) satisfies the conditions of the Lagrange’s mean value theorem in \([x_1, x_2]\). Therefore there exists one point \( c \) in \([x_1, x_2]\) such that

\[
\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0
\]

\[
\Rightarrow f(x_2) - f(x_1) = (x_2 - x_1)f'(c) > 0 \text{ (by (ii) of the hypothesis)}
\]

\[
\Rightarrow f(x_2) > f(x_1)
\]

Thus for \( x_1 < x_2 \) we have \( f(x_1) < f(x_2) \) i.e. the function is increasing in \([a, b]\).

**Corollary 4:** If a function \( f \) defined on \([a, b]\) be such that

(i) \( f \) is continuous on \([a, b]\)

(ii) \( f'(x) < 0 \forall x \in [a, b] \),

then \( f \) is strictly decreasing in \([a, b]\).

**Proof:** Similar to corollary 3.

**Example 9:** Verify Lagrange’s mean value theorem for the function

\[
f(x) = x(x-1)(x-2) \text{ in } [0, \frac{1}{2}]
\]

**Solution:** Since \( f \) is a polynomial in \( x \), so \( f \) is continuous on \([0, \frac{1}{2}]\) and derivable on \([0, \frac{1}{2}]\). Therefore by Lagrange’s mean value theorem \( \exists a \ c \in [0, \frac{1}{2}] \) such that

\[
f'(c) = \frac{f\left(\frac{1}{2}\right) - f(0)}{\left(\frac{1}{2}\right) - 0}
\]

\[
\Rightarrow \left(\frac{1}{2}\right) f'(c) = f\left(\frac{1}{2}\right) - f(0)
\]

\[
= \left(\frac{1}{2}\right) \left(\frac{1}{2} - 1\right) \left(\frac{1}{2} - 2\right) - 0
\]

\[
= \frac{3}{8}
\]

\[
\Rightarrow f'(c) = \frac{3}{4}
\]

Now \( f'(x) = (x-1)(x-2) + x(x-2) + x(x-1) = 3x^2 - 6x - 2 \)

\[
\Rightarrow f'(c) = \frac{3}{4} \Rightarrow 3c^3 - 6c - 2 = \frac{3}{4}
\]
\[ \Rightarrow c = \frac{6 \pm \sqrt{21}}{6} \]

But \( c = \frac{6 + \sqrt{21}}{6} \notin [0, \frac{1}{2}] \) and \( c = \frac{6 - \sqrt{21}}{6} \in [0, \frac{1}{2}] \) i.e. the theorem is valid.

2.3.2. Geometrical significance of Lagrange’s Mean Value Theorem

Interpreting each condition of the hypothesis geometrically, the Lagrange’s Mean Value Theorem states:

Let a curve \( y = f(x) \) be drawn continuously between two points \( a \) and \( b \) such that the tangents can be drawn at every point between \( a \) and \( b \). Let \( A(a, f(a)) \) and \( B(b, f(b)) \) be two points on this curve then there exists at least one point \( c \) between \( A \) and \( B \), \( c \neq a, c \neq b \) i.e. \( c \in ]a, b[ \) at which the tangent drawn to the curve is parallel to the chord \( A \) (fig 2.2).
From the figure 2.2, it is evident that there may exist more than one point between a and b where the tangents drawn are parallel to the chord joining A (a, f(a)) and B (b, f(b)).

**Remark 4:** Lagrange’s mean value theorem does not hold even if one condition of the hypothesis is not true:

Examples 2 and 3 of remark 2 of section 2 can be verified for Lagrange’s mean value theorem also. In both cases the theorem is not valid.

**Remark 5:** There may be some functions for which one or both conditions of the hypothesis of Lagrange’s theorem are not true but still a point c can be obtained for which the conclusion of the theorem holds true. In other words we say that the conditions of the Lagrange’s theorem are only sufficient but not necessary for the conclusion. For example:

**Example 10:** Consider a function

\[ f(x) = \begin{cases} 
0, & 0 \leq x < \frac{1}{4} \\
x, & \frac{1}{4} \leq x < \frac{1}{2} \\
\left(\frac{x}{2}\right) + 1, & \frac{1}{2} \leq x \leq 2 
\end{cases} \]

Verify that the function f is neither continuous in [0,2] nor derivable in ]0,2[ but still at the point \( x = \frac{1}{2} \), the conclusion of the theorem holds.

**Remark 6: Alternative form of Lagrange’s Mean Value Theorem**

Let us take \( h = b - a \). Then the interval [a, b] becomes [a, a + h]. Any element which lies in this interval is of the form \( a + \theta h \), where \( 0 < \theta < 1 \). Hence Lagrange’s theorem can be restated as follows:
Let \( f \) be a function defined on \([a, a+h]\) such that

(i) \( f \) is continuous in \([a, a+h]\)

(ii) \( f \) is derivable in \([a, a+h]\)

Then there exists at least one real \( 0 < \theta < 1 \), such that

\[
f'(a+\theta h) = \frac{f(a+h) - f(a)}{h}
\]

equivalently, \( f(a+h) = f(a) + h f'(a+\theta h) \).

Example 11: Find the number \( \theta \) that appears in the conclusion of Lagrange’s mean value theorem for the function \( f(x) = x^2 - 2x + 3 \) when \( a=1 \) and \( h = \frac{1}{2} \).

Solution: The function \( f \), being polynomial in \( x \), is continuous in \([a, a+h]\) i.e. in \([1, 1 + \frac{1}{2}] \) i.e. in \([1, 3/2]\) and derivable in \([1, 3/2]\). Therefore, by Lagrange’s mean value theorem, \( \exists \) at least one \( 0 < \theta < 1 \), satisfying

\[
f(a + h) = f(a) + h f'(a + \theta h)
\]

i.e. \( f(3/2) = f(1) + \frac{1}{2} f'(1 + \theta/2) \)

\[\Rightarrow \quad 9/4 = 2 + \frac{1}{2} \{ 2(1 + \theta/2) - 2 \} \]

\[\Rightarrow \quad 9/4 = 2 + \theta/2 \Rightarrow \theta = \frac{1}{2} \]

Example 13: Prove that for any quadratic function \( px^2 + qx + r \), the value of \( \theta \) in Lagrange’s theorem is always \( \frac{1}{2} \) whatever \( p, q, r, a, h \) may be.

Solution: Let \( f(x) = px^2 + qx + r \) and the interval is \([a, a+h]\).

Since \( f \) is a polynomial so \( f \) is continuous in \([a, a+h]\) and derivable in \([a, a+h]\). So by Lagrange’s theorem value theorem \( \exists \) at least one \( 0 < \theta < 1 \), satisfying

\[
f(a + h) = f(a) + h f'(a + \theta h)
\]

i.e. \( (a + h)^2 + q (a + h) + r = p a^2 + q a + r + h ( 2 p (a + \theta h) + q ) \)

\[\Rightarrow \quad p \{(a + h)^2 - a^2 \} + q h = 2aph + 2p\theta h^2 + q h \]

\[\Rightarrow \quad p\{h (2a + h)\} = 2aph + 2p\theta h^2 \]
Since \( \theta \) is independent of \( p, q, r, a, h \). So the value of \( \theta \) is always \( \frac{1}{2} \), whatever the values of these constants are.

**Example 14:** Separate the intervals in which the function
\[ f(x) = x^3 - 6x^2 + 9x + 1 \]
is increasing and decreasing.

**Solution:** We have \( f'(x) = 3x^2 - 12x + 9 \)

By corollary 3, if \( f'(x) > 0 \) \( \forall x \), then the function increases.

\[
\text{i.e. if } 3x^2 - 12x + 9 > 0 \Rightarrow x^2 - 4x + 3 > 0 \text{ i.e. } (x - 1)(x - 3) > 0
\]

\[
\Rightarrow \text{either } (x - 1) > 0 \text{ and } (x - 3) > 0
\]
\[
\text{or } (x - 1) < 0 \text{ and } (x - 3) < 0
\]

\[
\Rightarrow \text{either } x > 1 \text{ and } x > 3 \text{ or } x < 1 \text{ and } x < 3
\]

\[
\Rightarrow \text{either } x > 3 \text{ or } x < 1 \Rightarrow x \in [3, \infty \text{ or } x \in ]-\infty, 1]
\]

Thus function increases in \([3, \infty \text{ and } ]-\infty, 1]\)

For a function to be decreasing, by corollary 4, \( f'(x) = 3x^2 - 12x + 9 < 0 \)

Proceeding as above, we have

\[
\text{either } (x - 1) > 0 \text{ and } (x - 3) < 0
\]
\[
\text{or } (x - 1) < 0 \text{ and } (x - 3) > 0
\]

i.e. \( f \) decreases in \([1,3]\).

**Example 14:** Let \( f \) be defined and continuous in \([a - h, a + h]\) and derivable in \([a - h, a + h]\). Prove that there exists a real number \( \theta \), \( 0 < \theta < 1 \), for which
\[ f(a + h) - 2f(a) + f(a - h) = h(f'(a+\theta h) - f'(a - \theta h)) \]

**Solution:** Let us define a function \( F \) as follows:
\[ F(t) = f(a + h t) + f(a - ht) \text{ on } [0,1] \]
As \( f \) is given to be continuous on \( [a – h, a + h] \) and derivable in \( ]a – h, a + h[ \), therefore the function \( F \) is also continuous in \( [0,1] \) and derivable in \( ]0,1[ \). So, by Lagrange’s theorem, there exists a \( \theta \), \( 0 < \theta < 1 \), such that

\[
F(1) – F(0) = F'(\theta)
\]

\[
\Rightarrow f(a + h) – f(a – h) – 2f(a) = h\{f'(a+\theta h) – f'(a - \theta h)}
\]

**Example 15:** Use Lagrange’s mean value theorem to show that:

\[
x (1 + x^2)^{-1} < \tan^{-1} x < x \quad \forall x > 0
\]

**Solution:** Consider the function

\[
f(x) = \tan^{-1} x \quad \text{in} \quad [0, x]
\]

Since \( \tan^{-1} x \) is continuous for all \( x > 0 \), so \( f \) is continuous in \([0, x]\).

Also \( f'(x) = (1+ x^2)^{-1} \) which is defined for all \( x > 0 \), so \( f \) is derivable in \([0, x]\).

Thus by Lagrange’s mean value theorem \( \exists \) a real number \( c \in ]0, x[ \) such that

\[
f(x) – f(0) = (x – 0) f'(c)
\]

\[
\Rightarrow \tan^{-1} x – \tan^{-1} 0 = x (1+ c^2)^{-1}
\]

\[
\Rightarrow \tan^{-1} x – 0 = x / (1 + c^2) \quad \text{.............................................(I)}
\]

Now \( c \in ]0, x[ \) implies \( 0 < c < x \quad \Rightarrow \quad 1 + c^2 < 1+ x^2 \)

\[
\Rightarrow \quad (1 + x^2)^{-1} < (1+ c^2)^{-1}
\]

\[
\Rightarrow \quad x (1 + x^2)^{-1} < x (1+ c^2)^{-1} \quad \text{(multiplying both sides by \( x > 0 \))}
\]

\[
\Rightarrow \quad x (1 + x^2)^{-1} < \tan^{-1} x \quad \text{(by (I))} \quad \text{.............................................(II)}
\]

Also \( 0 < c \Rightarrow 1 < 1+ c^2 \Rightarrow (1+ c^2)^{-1} < 1 \)

\[
\Rightarrow \quad x (1+ c^2)^{-1} < x \quad \text{(multiplying both sides by \( x > 0 \))}
\]

\[
\Rightarrow \quad \tan^{-1} x < x \quad \text{(by (I))} \quad \text{.............................................(III)}
\]

(II) and (III) establishes the result.
Exercise 2.2

1. Verify Lagrange’s Mean value theorem for the following functions:
   (i) \( f(x) = x(x-2) \) in \([1,2]\)
   (ii) \( f(x) = |x| \) in \([0,2]\)
   (iii) \( f(x) = \cos x \) in \([0,\pi/2]\)
   (iv) \( f(x) = x^a \) , \( a \) being a positive integer, in \([-1,1]\)
   (v) \( f(x) = \log x \) in \([1,e]\)

2. Examine the validity of the conclusion of the Lagrange’s Mean value theorem for the following functions:
   (i) \( f(x) = 1 - (x - 1)^{2/3} \) in \([0,2]\)
   (ii) \( f(x) = \begin{cases} 
   x^2, & 1 < x < 2 \text{ in } [1,2] \\
   1, & x = 2 
   \end{cases} \)
   (iii) \( f(x) = 1/x \) in \([1,4]\)

3. Let \( f \) be defined and continuous in \([a - h, a + h]\) and derivable in \([a - h, a + h]\). Prove that there exists a real number \( \theta \), \( 0 < \theta < 1 \), for which
   \[ f(a + h) - f(a - h) = h[f'(a + \theta h) + f'(a - \theta h)] \]

4. Verify that on the curve \( y = f(x) \) where \( f(x) = ax^2 + bx + c \) (\( a, b, c \) being real constants, \( a \neq 0 \)), the chord joining the points \((p, f(p))\) and \((q, f(q))\) is parallel to the tangent at the point \( x = \frac{1}{2}(p + q) \).

5. Using Lagrange’s mean value theorem, show that for \( \forall x > 0 \)
   (i) \( e^x > 1 + x + x^2 \)
   (ii) \( e^{-x} > 1 - x \)
\[ (iii) \quad x - \frac{x^2}{3} < \tan^{-1} x \]
\[ (iv) \quad x - \frac{x^3}{6} < \sin x < x \]

2.4 Cauchy’s Mean Value Theorem

2.4.1 Statement: Let \( f \) and \( g \) be two functions defined on \([a, b]\) such that

\( (i) \) \( f \) and \( g \) are continuous in \([a, b]\)
\( (ii) \) \( f \) and \( g \) are differentiable in \( ]a, b[\)
\( (iii) \) \( g'(x) \neq 0 \quad \forall \; x \in ]a, b[ \)

Then there exists at least one point \( c \) in \([a, b]\) such that

\[
\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}
\]

Proof: We, at first, claim that because of condition (iii) of the statement, \( g(a) \neq g(b) \). For, if \( g(a) = g(b) \), then in view of the conditions (i) and (ii) of the hypothesis, the function \( g \) satisfies all conditions of Rolle’s theorem and as a result there exists at least one \( p \in ]a, b[ \) such that \( g'(p) = 0 \), which contradicts the condition (iii). Hence the claim.

Define a function \( F \) as

\[ F(x) = f(x) + Ag(x) \quad \forall \; x \in [a, b], \]
where \( A \) is a constant to be chosen suitably later.

Because of the continuity of the functions \( f \) and \( g \) in \([a, b]\) (by (i)) and the derivability of \( f \) and \( g \) in \( ]a, b[ \) (by (ii)), the function \( F \) is continuous in \([a, b]\) and derivable in \( ]a, b[ \). Choose \( A \) such that \( F(a) = F(b) \).

\[ \text{i.e. } f(a) + Ag(a) = f(b) + Ag(b) \]

\[ \text{i.e. } -A = \frac{f(b) - f(a)}{g(b) - g(a)} \]

\[ \text{...............} \]...(I)
Since $F$ satisfies all conditions of Rolle’s theorem, therefore \( \exists \) at least one real number \( c \in ]a, b[ \) such that \( F'(c) = 0 \)

\[
\Rightarrow f'(c) + Ag'(c) = 0
\]

\[
\Rightarrow -A = \frac{f'(c)}{g'(c)} \quad \text{…………………………………………………………(II)}
\]

Combining (I) and (II) we get:

\[
\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}
\]

**Remark 7: Alternative form of Cauchy’s Mean Value Theorem**

*Let $f$ and $g$ be two functions defined on $[a, a + h]$ such that*

(i) $f$ and $g$ are continuous in $[a, a + h]$

(ii) $f$ and $g$ are derivable in $]a, a + h[$

(iii) $g'(x) \neq 0 \ \forall \ x \in ]a, a + h[$

*Then there exists at least one point $\theta$, $0 < \theta < 1$ such that*

\[
\frac{f(a + h) - f(a)}{g(a + h) - g(a)} = \frac{f'(a + \theta h)}{g'(a + \theta h)}
\]

**Remark 8:** In Cauchy’s mean value theorem, if we choose $g(x) = x$, then we can deduce Lagrange’s theorem. Therefore, we consider Lagrange’s theorem as a particular case of Cauchy’s theorem.

**Remark 9:** In Cauchy’s mean value theorem, if we choose $g(x) = 1$, then we can deduce Rolle’s theorem.

**Example 15:** Verify the Cauchy’s theorem for the functions

\( f(x) = \sin x, \ g(x) = \cos x \) in the interval \([- \pi/2, 0]\)
Solution: Obviously, f and g are continuous in $[-\pi/2, 0]$ and derivable in $]-\pi/2, 0[$. Also $g'(x) = -\sin x \neq 0 \ \forall \ x \in ]-\pi/2, 0[$. Hence by Cauchy’s theorem

$$\exists \ c \in ]-\pi/2, 0[ \text{ such that }$$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\Rightarrow \frac{\cos c}{-\sin c} = \frac{\sin (0) - \sin (-\pi/2)}{\cos (0) - \cos (-\pi/2)}$$

$$\Rightarrow \frac{\cos c}{-\sin c} = \frac{0 + 1}{1 - 0}$$

i.e. $\tan c = -1$ i.e. $c = -\pi/4 \in ]-\pi/2, 0[$

Thus Cauchy’s theorem is valid.

2.4.2. Geometrical interpretation of Cauchy’s Mean Value Theorem:

The conclusion of the Cauchy’s theorem gives:

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

equivalently,

$$\frac{f'(c)}{g'(c)} = \frac{(f(b) - f(a)) / (b - a)}{(g(b) - g(a)) / (b - a)}$$

But, from our earlier knowledge of calculus, we know that $f'(c)$ represents the slope of the tangent to the curve $y = f(x)$ at the point $(c, f(c))$ and $g'(c)$ represents the slope of the tangent to the curve $y = g(x)$ at the point $(c, g(c))$. Also, $\frac{f(b) - f(a)}{(b - a)}$ is the slope of the chord joining the points $(a, f(a))$ and $(b, f(b))$ and
\[
\frac{g(b) - g(a)}{(b - a)}\text{ is the slope of the chord joining the points (a, g(a)) and (b, g(b)).}
\]

Thus, geometrically, the Cauchy's Mean Value Theorem can be stated as:

*If two curves \( y = f(x) \) and \( y = g(x) \) are drawn continuously between two points with ordinates \( x = a \) and \( x = b \) (condition(i)) such that tangents can be drawn to each of the curves at each point lying between \( x = a \) and \( x = b \) (condition(ii)) and nowhere the tangent to the curve \( y = g(x) \) is parallel to x-axis (condition(iii)), then there exists at least one point \( c \) between \( a \) and \( b \) such that the ratio of the slopes of the chords joining \((a, f(a)), (b, f(b))\) and \((a, g(a)), (b, g(b))\) is equal to the ratio of the slopes of the tangents to the curves at \( x = c \) (fig. 2.3).*
Exercise 2.3

1. Verify Cauchy’s theorem for the following functions in the mentioned interval:

   (i) \( f(x) = x^2, \ g(x) = x^4, \ a = 2, \ b = 4. \)

   (ii) \( f(x) = x^2, \ g(x) = x, \ a = 0, \ b = 1. \)

   (iii) \( f(x) = e^x, \ g(x) = e^{-x}, \ a = 0, \ b = 1. \)

   (iv) \( f(x) = x(x - 1)(x - 2), \ g(x) = x(x - 2)(x - 3), \ a = 0, \ b = \frac{1}{2}. \)

2. Show that

\[
\frac{\sin \alpha - \sin \beta}{\cos \alpha - \cos \beta} = \cot \theta, \text{ where } 0 < \alpha < \theta < \beta < \pi/2
\]

(Hint: In the solution of the example 15, choose the interval as \([\alpha, \beta]\) instead of \([-\pi/2, 0]\).)

2.5 Taylor’s Theorem

2.5.1 Statement: If a function \( f \) defined on \([a, b]\) is such that

   (i) the \((n-1)\)th derivative \( f^{(n-1)} \) is continuous in \([a, b] \)

   (ii) the \((n-1)\)th derivative \( f^{(n-1)} \) is derivable in \([a, b]\),

then there exists a point \( c \in ]a, b[ \) such that

\[
f(b) = f(a) + (b - a) f'(a) + \frac{(b - a)^2}{2!} f''(a) + \ldots + \frac{(b - a)^{n-1}}{(n - 1)!} f^{(n-1)}(a) + R_n
\]

where \( R_n = \frac{(b - a)^p}{p(n-1)!} f^{(n)}(c), p \) being a positive integer.

Proof: By hypothesis, the function \( f \) is such that derivatives of all order exist and are continuous. This means that \( f, f', f'', \ldots, f^{(n-1)} \) are all continuous in \([a, b]\) and derivable in \([a, b]\) ………………………………………………………………..(I)

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Consider a function $F$ defined as

$$F(x) = f(b) - f(x) - (b - x) f'(x) - \frac{(b - x)^2}{2!} f''(x) - \frac{(b - x)^3}{3!} f'''(x) - \ldots$$

$$- \frac{(b - x)^{n-1}}{(n-1)!} f^{(n-1)}(x) - A \frac{(b - x)^p}{(b - a)^p}$$

where the constant $A$ is such that $F(a) = F(b)$………………….(II)

Thus $F$ satisfies the conditions of Rolle’s theorem. Therefore, then there exists a point $c \in ]a, b[$ such that $F'(c) = 0$.

i.e. $0 - f'(c) - \{(b - c) f''(c) - f'(c)\} - \left\{\frac{(b - c)^2}{2!} f'''(c) - 2 \cdot \frac{(b - c)}{2!} f''(c)\right\}$

$$- \ldots - \frac{(b - c)^{p-1}}{(n-1)!} f^{(n)}(c) - A p \frac{(b - c)^{p-1}}{(b - a)^p} = 0$$

i.e. $A = \frac{(b - c)^{n-p} (b - a)^p}{p(n-1)!} f^{(n)}(c) ............................................(III)$

Also $F(a) = F(b)$ gives

$$f(b) - f(a) - (b - a) f'(a) - \frac{(b - a)^2}{2!} f''(a) - \frac{(b - a)^3}{3!} f'''(a) - \ldots$$

$$- \frac{(b - a)^{n-1}}{(n-1)!} f^{(n-1)}(a) = A ......................................................(IV)$$

Substituting for $A$ from (III) in (IV) we get the result.

**Remark 1:** The expression

$$R_n = \frac{(b - a)^p (b - c)^{n-p}}{p(n-1)!} f^{(n)}(c)$$

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is called the *Taylor's Remainder after n terms*.

**Remark 2: Alternative form of Taylor’s Theorem**

As discussed in case of Lagrange’s theorem, if we choose $b = a + h$ then, alternately Taylor’s theorem can be stated as follows:

*If a function $f$ defined on $[a, a + h]$ be such that*

1. *the $(n−1)$th derivative $f^{(n−1)}$ is continuous in $[a, a + h]$*
2. *the $(n−1)$th derivative $f^{(n−1)}$ is derivable in $(a, a + h)$,*

*then there exists at least one point $\theta$, $0 < \theta < 1$ such that*

$$f(a + h) = f(a) + h f'(a) + \sum_{p=2}^{n} \frac{h^{p−1}}{(n−1)!} f^{(n−1)}(a) + R_n$$

*where $R_n = \frac{h^n (a + h - a + \theta h)^{n-p}}{p(n−1)!} f^{(n)}(a + \theta h)$, $p$ being a positive integer.*

**Remark 3:** If $p = n$, the Taylor’s remainder $R_n$ becomes

$$R_n = \frac{(b - a)^n}{n!} f^{(n)}(c)$$

*or in the notations of remark 2*

$$R_n = \frac{h^n}{n!} f^{(n)}(a + \theta h)$$

*and is called the *Lagrange’s form of Remainder after n terms*. *

**Remark 4:** If $p = 1$, the Taylor’s remainder becomes

$$R_n = \frac{(b - a)(b - c)^{n−1}}{(n−1)!} f^{(n)}(c)$$

*or in the notations of remark 2*
\[ R_n = \frac{h^n (1-0)^{n-1}}{(n-1)!} f^{(n)}(a + \theta h) \]

and is called the **Cauchy’s form of Remainder after n terms.**

**Remark 5:** If in the alternate statement of the remark 2, we put \( a + h = x \) and \( a = 0 \) then we get the **Maclaurin’s Theorem** as stated below:

If a function \( f \) defined on \([0, x]\) be such that

(i) the \((n-1)\)th derivative \( f^{(n-1)} \) is continuous in \([0, x]\)

(ii) the \((n-1)\)th derivative \( f^{(n-1)} \) is derivable in \([0, x]\),

then there exists at least one point \( \theta \), \( 0 < \theta < 1 \) such that

\[ f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n \]

where \( R_n = \frac{x^n}{n!} f^{(n)}(\theta x) \) (Lagrange’s remainder after \( n \) terms)

\[ R_n = \frac{x^n (1-\theta)^{n-1}}{(n-1)!} f^{(n)}(\theta x) \] (Cauchy’s remainder after \( n \) terms).

Using this remark 5, we shall find out expansions of different functions. For this we will use the following result:

2.5.5: **Maclaurin’s infinite series expansion:**

**Statement:** If a function \( f \) defined on \([0, h]\) is such that

(i) \( f^{(n)} \) exists for each \( n \) in \([0, h]\)

(ii) Taylor’s remainder \( R_n \to 0 \) as \( n \to \infty \)

then for each \( x \in [0, h] \)
\[ f(x) = f(0) + x \frac{f'(0)}{2!} + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^n(0) + \ldots \]

This is called Maclaurin’s infinite series expansion.

2.6 Power series expansion of some standard functions:

2.6.1 Maclaurin’s infinite series expansion of \( e^x \):

Let \( f(x) = e^x \). Then

\[ f'(x) = e^x, \ f''(x) = e^x, \ldots, \ f^{(n)}(x) = e^x \] for all \( n \in \mathbb{N} \)

i.e. \( f \) has derivatives of all order.

Also

\[ f'(0) = f''(0) = \ldots = f^{(n)}(0) = e^0 = 1 \] for all \( n \in \mathbb{N} \)

The Lagrange’s form of remainder after \( n \) terms is

\[ R_n = \frac{x^n}{n!} f^{(n)}(\theta x), \quad 0 < \theta < 1 \]

i.e. \( R_n = \frac{x^n}{n!} e^{\theta x}, \quad 0 < \theta < 1 \) (refer to section III for nth derivative of \( e^x \))

\[ \Rightarrow \lim_{n \to \infty} R_n = \lim_{n \to \infty} \frac{x^n}{n!} e^{\theta x} = 0 \]

Thus all conditions of Maclaurin’s expansion are satisfied. Hence

\[ f(x) = f(0) + x \frac{f'(0)}{2!} + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^n(0) + \ldots \]

i.e \( e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \) for all \( x \) in \( \mathbb{R} \).
2.6.2 Maclaurin’s infinite series expansion of sin x:

Let \( f(x) = \sin x \). Then

\[
f'(x) = \sin x, \quad f''(x) = \cos x, \quad \ldots, \quad f^{(n)}(x) = \sin (x + n\pi/2) \quad \text{for all } n \in \mathbb{N}
\]

i.e. \( f \) has derivatives of all order.

Also

\[
f'(0) = 0, \quad f''(0) = 1, \quad f^{(3)}(0) = 0, \quad f^{(4)} = -1 \quad \ldots \ldots
\]

The Lagrange’s form of remainder after \( n \) terms is

\[
R_n = \frac{x^n}{n!} \quad f^{(n)}(\theta x), \quad 0 < \theta < 1
\]

i.e.

\[
R_n = \frac{x^n}{n!} \quad \sin (\theta x + n\pi/2), \quad 0 < \theta < 1
\]

\[
\Rightarrow \lim_{n \to \infty} \left| R_n \right| = \lim_{n \to \infty} \left| \frac{x^n}{n!} \sin (\theta x + n\pi/2) \right| = 0
\]

because \( \left| \sin y \right| \leq 1 \) for all \( y \) and hence for \( y = (\theta x + n\pi/2) \).

Hence, by Maclaurin’s theorem,

\[
f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots \ldots
\]

\[
\sin x = 0 + x + 0 + \frac{x^3}{3!} (-1) + \ldots \ldots
\]

i.e.

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \ldots \quad \text{for all } x \in \mathbb{R}
\]

Similarly try for \( \cos x \), you will get:

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \ldots \quad \text{for all } x \in \mathbb{R}
\]
2.6.3 Maclaurin’s expansion for \((1 + x)^m\), \(m\) being a positive integer:

Let \(f(x) = (1 + x)^m\), \(m\) being a positive integer. Then

\[
\begin{align*}
    f'(x) &= m(1 + x)^{m-1}, \\
    f''(x) &= m(m-1)(1 + x)^{m-2} \\
    f^{(m)}(x) &= m(m - 1)(m - 2) \cdots \cdot 3 \cdot 2 \cdot 1 = m! \\
    f^{(n)}(x) &= 0 \text{ for all } n \geq m
\end{align*}
\]

i.e. \(f\) has derivatives of all order.

Also

\[
\begin{align*}
    f'(0) &= m, \\
    f''(0) &= m(m-1), \quad \ldots \\
    f^{(m)}(0) &= m!, \\
    f^{(n)}(0) &= 0 \text{ for all } n \geq m
\end{align*}
\]

The Lagrange’s form of remainder after \(n\) terms is

\[
R_n = \frac{x^n}{n!} f^{(n)}(\theta x), \quad 0 < \theta < 1
\]

\[\Rightarrow R_n \to 0 \text{ as } n \to \infty\]

Hence, by Maclaurin’s theorem,

\[
f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots \ldots 
\]

i.e. for all \(x\) in \(\mathbb{R}\)

\[
(1 + x)^m = 1 + m x + \frac{m(m-1)}{2!} x \ldots + \frac{m(m - 1)(m - 2)}{3!} x^3 + \ldots + x^m
\]

2.6.4 Maclaurin’s infinite series expansion of \(\log(1 + x)\) for \(-1 < x \leq 1\)

Let \(f(x) = \log(1 + x)\). Then

\[
\begin{align*}
    f'(x) &= \frac{1}{1 + x}, \quad f''(x) = -\frac{1}{(1 + x)^2}, \quad \ldots 
\end{align*}
\]

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\[ f^{(n)}(x) = \frac{(-1)^{n-1} (n-1)!}{(1 + x)^n} \text{ for all } n \in \mathbb{N} \text{ and for all } x > -1 \]

i.e. \( f \) has derivatives of all order.

Also \( f(0) = \log 1 = 0, f'(0) = 1, f''(0) = -1, \ldots \)

We shall consider two cases:

**Case 1**: \( 0 \leq x \leq 1 \)

Then, the Lagrange’s form of remainder after \( n \) terms is

\[
R_n = \frac{x^n}{n!} f^{(n)}(\theta x), \quad 0 < \theta < 1
\]

i.e. \( R_n = \frac{x^n}{n!} \frac{(-1)^{n-1} (n-1)!}{(1 + \theta x)^n} \) for \( 0 < \theta < 1 \)

i.e. \( R_n = \frac{(-1)^{n-1}}{n} \left( \frac{x}{(1 + \theta x)} \right)^n \)

Since \( \frac{x}{(1 + \theta x)} < 1 \) and \( (1/n) \to 0 \) as \( n \to \infty \)

Therefore \( R_n \to 0 \) as \( n \to \infty \)

**Case 2**: \( -1 < x < 0 \)

The Cauchy’s form of remainder after \( n \) terms is

\[
R_n = \frac{x^n}{(n-1)!} (1 - \theta)^{n-1} f^{(n)}(\theta x)
\]

\[
= (-1)^{n-1} x^n \left( \frac{1 - \theta}{(1 + \theta x)} \right)^{n-1} \frac{1}{(1 + \theta x)}
\]
Since $-1 < x$ and $0 < 0 < 1 \Rightarrow -0 < 0x \Rightarrow 1 - 0 < 1 + 0x$

\[
\frac{1 - 0}{1 + 0x} < 1 \quad \text{so that} \quad \left(\frac{1 - 0}{1 + 0x}\right)^{n-1} \rightarrow 0 \text{ as } n \rightarrow \infty
\]

Also for $x < 0 \Rightarrow x^n \rightarrow 0 \text{ as } n \rightarrow \infty$

Hence $R_n \rightarrow 0$ as $n \rightarrow \infty$

Combining case 1 and case 2 we have $R_n \rightarrow 0$ as $n \rightarrow \infty$ for $-1 < x \leq 1$

Hence, the Maclaurin’s series is:

\[
f(x) = f(0) + x \frac{f'(0)}{2!} + \frac{x^2}{3!} f''(0) + \cdots + \frac{x^n}{n!} f^n(0) + \cdots \quad \text{(I)}
\]

That is, for all $x$ such that $-1 < x \leq 1$

\[
\log(1 + x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots
\]

Example 15: Assuming the validity of expansion, find the series expansion of $f(x) = e^{2x}$ for all real values of $x$.

Solution: Since the validity is assumed, so the Maclaurin’s expansion gives

\[
f(x) = f(0) + x \frac{f'(0)}{2!} + \frac{x^2}{3!} f''(0) + \cdots + \frac{x^n}{n!} f^n(0) + \cdots \quad \text{(I)}
\]

We know that,

\[
f(0) = e^0 = 1
\]

and \[f^{(n)}(x) = 2^n e^{2x}\]

\[
\Rightarrow f'(0) = 2, \ f''(0) = 2^2 = 4, \ \text{and} \quad f^{(n)}(0) = 2^n
\]
Substituting in (I), we get:

\[
e^{2x} = 1 + 2x + 2^2 \frac{x^2}{2!} + \ldots + 2^n \frac{x^n}{n!} + \ldots
\]

\[
e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \ldots + \frac{2^n x^n}{n!} + \ldots
\]

Exercise 2.4

1. Assuming the possibility of the expansion, prove that for all \( x \) in \( \mathbb{R} \)

   (i) \( e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} - \ldots \)

   (ii) \( \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \)

   (iii) \( (1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots \)

2. For \(- 1 < x \leq 1\), show that

   \( \log (1 - x) = -x - \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!} - \ldots \)
Lesson 3

INDETERMINATE FORMS

3.1 Introduction:
Let \( f(x) \) and \( g(x) \) be two functions defined on an interval \( I \).

If \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) both exist and \( \lim_{x \to a} g(x) \neq 0 \) then

\[
\lim_{x \to a} \frac{f(x)}{g(x)} \quad \text{exists.}
\]

We also know that if \( \lim_{x \to a} f(x) \) as \( x \to a \) is finite and non zero then, even if \( \lim g(x) = 0 \), the limit of the ratio \( f(x)/g(x) \) as \( x \to a \) exists and is equal to zero. But what will happen if \( \lim f(x) \) as \( x \to a \) is infinite or zero? In that case this ratio is called indeterminate form. Precisely, we may define it as follows:

**Definition:** The ratio \( \frac{f(x)}{g(x)} \) is said to represent the indeterminate form \( \frac{0}{0} \) as \( x \to a \) if

\[
\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0
\]

This lesson discusses the ways to find the limit of this indeterminate form successfully. Besides this we may like to discuss a few more types of indeterminate forms like \( \infty / \infty \), \( 0/\infty \), \( \infty - \infty \), \( 1^\infty \), \( \infty^0 \) etc. We shall state a number of results to evaluate these ratios but proofs are not to be done at this stage.

3.2: L’ Hopital’s Rule for \( \frac{0}{0} \) form

**Statement:** Let \( f(x) \) and \( g(x) \) be two functions defined on an interval \( I \) such that
(i) \( \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 \)

(ii) \( f'(x) \) and \( g'(x) \) exist and \( g(x) \neq 0 \) and \( g'(x) \neq 0 \) for all \( x \in ]a-\delta, a+\delta[ \), \( \delta > 0 \) except possibly at \( x = a \)

\[
f'(x)
\]

(iii) \( \lim_{x \to a} \frac{f'(x)}{g'(x)} \) exists

then \( \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \)

**Remark 1:** If \( \lim_{x \to a} \frac{f'(x)}{g'(x)} \) is again an indeterminate form, then

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)}
\]

Thus the rule can be generalized as follows:

If \( f(x) \) and \( g(x) \) are such that \( \lim_{x \to a} \frac{f^{(n-1)}(x)}{g^{(n-1)}(x)} \) represents the indeterminate form 0/0 and the functions \( f^{(n)}(x) \) and \( g^{(n)}(x) \) satisfy the conditions of the Theorem 3.1, then

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)} = \ldots = \lim_{x \to a} \frac{f^{(n)}(x)}{g^{(n)}(x)}
\]

3.3: L’Hospital’s Rule for \( \frac{\infty}{\infty} \)

**Statement:** Let \( f(x) \) and \( g(x) \) be two functions defined on an interval I such that
(i) \( \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty \)

(ii) \( f'(x) \) and \( g'(x) \) exist and \( g(x) \neq 0 \) and \( g'(x) \neq 0 \) for all \( x \in ]a-\delta, a+\delta[ \), \( \delta > 0 \) except possibly at \( x = a \)

(iii) \( \lim_{x \to a} \frac{f(x)}{g(x)} \) exists

then \( \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \)

**Remark 2:** Similar to remark 1, the theorem 3.2 can also be generalized to the higher order derivatives.

**Examples 1:** Evaluate \( \lim_{x \to 0} \frac{1 - \cos x}{x^2} \)

**Solution:** Let us write \( f(x) = 1 - \cos x \) and \( g(x) = x^2 \).

Now \( f(0) = 1 - \cos 0 = 1 - 1 = 0 \) and \( g(0) = 0 \)

\( f(x) \)

\( \therefore \lim_{x \to 0} \frac{f(x)}{g(x)} \) is of \( 0/0 \) form. Thus, using theorem 3.1, we have

\( \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} \)

\( \text{i.e.} \lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} \)

But as \( x \to 0, \sin x = 0 \) and \( 2x = 0 \)

\( \text{i.e.} \) we have, again, got a \( 0/0 \) form. So, using remark 1, we have
\[
\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{\cos 0}{2} = \frac{1}{2}
\]

**Examples 2:** Evaluate \(\lim_{x \to 0} \frac{\tan x - x}{x^2 \tan x}\)

**Solution:** Let us simplify the expression at first

\[
\lim_{x \to 0} \frac{\tan x - x}{x^2 \tan x} = \lim_{x \to 0} \left( \frac{\tan x - x}{x^3} \times \frac{x}{\tan x} \right)
\]

\[
= \lim_{x \to 0} \frac{\tan x - x}{x^3} \times \lim_{x \to 0} \frac{x}{\tan x}
\]

\[
= \lim_{x \to 0} \frac{\tan x - x}{x^3} \left( \frac{x}{\tan x} = 1 \right)
\]

\[
= \lim_{x \to 0} \frac{\tan x - x}{x^3} \left( \frac{0}{0} \text{ form} \right)
\]

\[
= \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} \left( \frac{0}{0} \text{ form} \right)
\]
\[
\begin{align*}
&= \lim_{x \to 0} \frac{2 \sec^2 x \tan x}{6x} \\
&= \lim_{x \to 0} \frac{2 \sec^2 x}{6} \times \lim_{x \to 0} \frac{\tan x}{x} \\
&= \frac{2 \sec^2 0}{6} = \frac{1}{3}
\end{align*}
\]

**Examples 3:** A student has evaluated the limit as follows:

\[
\lim_{x \to 1} \frac{x^3 + 3x - 4}{2x^2 - x - 1}
\]

\[
= \lim_{x \to 1} \frac{3x^2 + 3}{4x - 1} = \lim_{x \to 1} \frac{6x}{4} = \frac{3}{2}
\]

Do you agree with this answer? Give reasons.

**Solution:** No, we don’t agree with the solution done because

\[
\lim_{x \to 1} \frac{3x^2 + 3}{4x - 1} \text{ is not of } 0/0 \text{ form, rather its value is } -3
\]

therefore, the further application of L’ Hopital’s rule is not required.

**Remark 3:** As is evident from the example 3, we should check at each step, before using the L’ Hopital’s Rule that the limit of the expression is of the form 0/0, otherwise, you may get misleading results.

**Examples 4:** Evaluate

\[
\lim_{x \to \pi/2} \frac{\tan 3x}{\tan x}
\]
Solution: Since $\tan \left( \frac{3\pi}{2} \right) = \infty$ and $\tan(\pi/2) = \infty$, so we have, by theorem 3.2,
\[
\lim_{x \to \pi/2} \frac{\tan 3x}{\tan x} = \lim_{x \to \pi/2} \frac{3 \sec^2 3x}{\sec^2 x} = \lim_{x \to \pi/2} \frac{3 \cos^2 x}{\cos^2 3x} = \lim_{x \to \pi/2} \frac{-6 \sin x \cos x}{-6 \sin 3x \cos 3x} = \ldots \ldots \ldots (I)
\]
We know that $2 \sin A \cos A = \sin 2A$, therefore, (I) is
\[
= \lim_{x \to \pi/2} \frac{\sin 2x}{\sin 6x} = \lim_{x \to \pi/2} \frac{2 \cos 2x}{6 \cos 6x} = \frac{2 \cos (2\pi/2)}{6 \cos (6\pi/2)} = \frac{1}{3}
\]
Example 5: What should be the values of $a$ and $b$ if
\[
\lim_{x \to 0} \frac{x (1 - a \cos x) + b \sin x}{x^3} = \frac{1}{3}
\]
Solution: Since expression on the left hand side is of $0 / 0$ form, so using the theorem 3.1, we have
\[
\lim_{x \to 0} \frac{x (1 - a \cos x) + b \sin x}{x^3}
\]
\[
\lim_{x \to 0} \frac{1 - a \cos x + x (a \sin x) + b \cos x}{3x^2}
\]

For the further application of the L’ Hopital’s rule we should have limit of numerator as well as of denominator to be zero separately. Thus we have the first equation as

\[
1 - a \cos 0 + 0 (a \sin 0) + b \cos 0 = 0
\]

i.e.  \(1 - a + b = 0\)……………………………………………………………………………………..(I)

Assuming (I) to be true, we can apply L’ Hopital rule again. Hence we have

\[
\lim_{x \to 0} \frac{2a \sin x + x (a \cos x) - b \sin x}{6x}
\]

\[
= \lim_{x \to 0} \frac{ax \cos x + (2a - b) \sin x}{6x}
\]

\[
= \lim_{x \to 0} \frac{a \cos x - ax \sin x + (2a - b) \cos x}{6}
\]

\[
= \lim_{x \to 0} \frac{a \cos 0 - 0 (a \sin 0) + (2a - b) \cos 0}{6} = \frac{1}{3}
\]

i.e.  \(3a - b = 2\) …………………………………………………………………………………..(III)

Solving (I) and (III), we get \(a = 1/2\) and \(b = -1/2\).
Exercise 3.1

1. Evaluate:

(i) \( \lim_{x \to 0} \frac{x e^x - \log (1 + x)}{x^2} \)

(ii) \( \lim_{x \to 0} \frac{1 - \cos x^2}{x^2 \sin x^2} \)

(iii) \( \lim_{x \to 0} \frac{\tan x - x}{\sin x - x} \)

2. For what value of \( a \) does \( \lim_{x \to 0} \frac{\sin 2x + a \sin x}{x^2} \) is finite?

3. Evaluate:

(i) \( \lim_{x \to 0} \frac{\log x^2}{\cot x^2} \)

(ii) \( \lim_{x \to 0} \frac{x^2}{e^x} \)

(iii) \( \lim_{x \to 1} \frac{\log (x - 1) + \tan (\pi x/2)}{\cot \pi x} \)

(iv) \( \lim_{x \to (\pi/2)^+} \frac{\log (x - (\pi/2))}{\tan x} \)
3.4: Indeterminate forms of the type $0 \times \infty$

**Rule:** If \( \lim_{x \to a} f(x) = 0 \) and \( \lim_{x \to a} g(x) = \infty \) then \( \lim_{x \to a} [f(x)g(x)] \) is said to be of the form $0 \times \infty$.

In this case we rewrite the functions as

\[
\frac{f(x)}{1/g(x)} \quad \text{or} \quad \frac{g(x)}{1/f(x)}
\]

so that \( [f(x)g(x)] \) takes the form $0/0$ or $\infty/\infty$ as \( x \to a \) and hence can be evaluated using L’Hopital’s Rules.

**Example 3.6:** Show that \( \lim_{x \to 0} (1 - \cos x) \cot x = 0 \)

**Solution:** \( \lim_{x \to 0} (1 - \cos x) \cot x \) (is $0 \times \infty$ form, so by above rule)

\[
= \lim_{x \to 0} \frac{(1 - \cos x)}{\tan x} \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \quad \text{(form)}
\]

\[
= \lim_{x \to 0} \frac{\sin x}{\sec^2 x} = \frac{\sin 0}{\sec^2 0} = 0
\]

3.5: Indeterminate forms of the type $\infty - \infty$

If \( \lim_{x \to a} f(x) = \infty \) and \( \lim_{x \to a} g(x) = \infty \) then \( \lim_{x \to a} [f(x) - g(x)] \) is said to be of the form $\infty - \infty$.

To evaluate such limits, we have

**Rule:** We rewrite the functions as
\[
f(x) - g(x) = \frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x)g(x)}}
\]

so that \( \lim_{x \to a} \frac{f(x) - g(x)}{1} \) takes the usual \( 0/0 \) form.

**Example 3.7:** Evaluate \( \lim_{x \to 0} \cot x - \frac{1}{x^2} \)

**Solution:** As \( \lim_{x \to 0} \cot x = \infty \) and \( \lim_{x \to 0} \frac{1}{x^2} = \infty \)

\[
\therefore \text{We rewrite the expression to be evaluated as}
\]

\[
\lim_{x \to 0} \left( \cot^2 x - \frac{1}{x^2} \right) = \lim_{x \to 0} \left( \frac{\cos^2 x}{\sin^2 x} - \frac{1}{x^2} \right)
\]
\[
= \lim_{x \to 0} \frac{x^2\cos^2 x - \sin^2 x}{x^2\sin^2 x}
\]
\[
= \lim_{x \to 0} \frac{x^2\cos^2 x - \sin^2 x}{x^4} \times \frac{x^2}{\sin^2 x}
\]
\[
= \lim_{x \to 0} \frac{x^2\cos^2 x - \sin^2 x}{x^4} \quad \{ \text{because } \lim_{x \to 0} \frac{x^2}{\sin^2 x} = 1 \} \]
\[
= \lim_{x \to 0} \frac{x^2 (1 + \cos 2x) - (1 - \cos 2x)}{2x^4}
\]

\[
= \lim_{x \to 0} \frac{x^2 - 1 + (x^2 + 1) \cos 2x}{2x^4} \left( \frac{0 \text{ form}}{0} \right)
\]

\[
= \lim_{x \to 0} \frac{-(1 + x^2) \sin 2x + x + x \cos 2x}{4x^3} \left( \frac{0 \text{ form}}{0} \right)
\]

\[
= \lim_{x \to 0} \frac{-2(1 + x^2) \cos 2x - 6x \sin 2x + 1 + \cos 2x}{12x^2} \left( \frac{0 \text{ form}}{0} \right)
\]

\[
= \lim_{x \to 0} \frac{x^2 \sin 2x - 4x \cos 2x - \sin 2x}{6x} \left( \frac{0 \text{ form}}{0} \right)
\]

\[
= \lim_{x \to 0} \frac{10x \sin 2x - 4 \cos 2x + 2x^2 \cos 2x}{6} = \frac{2}{3}
\]

**Exercise 3.2**

1. Evaluate:

   (i) \( \lim_{x \to \pi/2} (1 - \sin x) \tan x \)

   (ii) \( \lim_{x \to \pi/2} \tan 3x \cot x \)
(iii) \( \lim_{x \to 0} x^2 \log x \)

(iv) \( \lim_{x \to 0} (a - x) \tan(\pi x/2a) \)

2. Evaluate:

(i) \( \lim_{x \to 0} \left( \frac{1}{x} - \frac{1}{\sin x} \right) \)

(ii) \( \lim_{x \to 0} \left( \frac{1}{x^2} - \frac{1}{\sin^2 x} \right) \)

(iii) \( \lim_{x \to 0} \left( \frac{1}{\log (x - 3)} - \frac{1}{x - 4} \right) \)

(iv) \( \lim_{x \to \pi/2} (\sec x - \tan x) \)

(v) \( \lim_{x \to 0} \left( \frac{1}{x} - \frac{1}{x^2} \log (1 + x) \right) \)

3.6: Indeterminate forms of the type \(0^\infty, 1^\infty, \infty^0, 0^0\)

If \( \lim_{x \to a} f(x) = 0 \) or \( 1 \) or \( \infty \) or \( 0 \) and \( \lim_{x \to a} g(x) = 0 \) or \( 1 \) or \( \infty \) or \( 0 \) respectively then

\( \lim_{x \to a} [f(x)]^{g(x)} \) takes one of the above forms. To evaluate this limit, we proceed as follows:

**Rule:** We write \( y = [f(x)]^{g(x)} \)

\[ \Rightarrow \log y = g(x) \log [f(x)] \]
\[ \lim_{x \to a} \log y = \lim_{x \to a} \{g(x) \log [f(x)]\} \]

The right hand limit can be evaluated by any of the above methods. Let this limit be \( L \). Then

\[ \lim_{x \to a} \log y = L \]

\[ \Rightarrow \lim_{x \to a} y = e^L \]

Hence, \( \lim_{x \to a} [f(x)]^{g(x)} = e^L \)

**Example 8:** Find \( \lim_{x \to 0^+} x^{(1/\log x)} \)

**Solution:** Since log function is not defined for negative values so we have restricted \( x \) up to \( 0+0 \) i.e only positive values are to be taken.

Here \( f(x) = x \) which tends to 0 as \( x \to 0^+ \) and \( g(x) = 1/\log x \) tends to \( \infty \) as \( x \to 0^+ \). So this function is of the form \( 0^\infty \).

Therefore, let

\[ y = x^{(1/\log x)} \]

\[ \Rightarrow \lim_{x \to 0^+} \log y = \lim_{x \to 0^+} (1/\log x) \log x \]

i.e. \( \log y = \lim_{x \to 0^+} \left( \frac{1}{\log x} \right) \log x = 1 \)

i.e. \( \log y = 1 \Rightarrow y = e \)
i.e. \( y = \lim_{x \to 0^+} x^{(1/\log x)} = e \)

**Example 3.9:** Evaluate \( \lim_{x \to 0} (\cos x)^{1/x^2} \)

**Solution:** It is of the form \( 1^\infty \).

Let \( y = (\cos x)^{1/x^2} \)

\[ \Rightarrow \lim_{x \to 0} \log y = \lim_{x \to 0} (1/x^2) \log \cos x \]

\[ = \lim_{x \to 0} \frac{\log \cos x}{x^2} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ form} \]

\[ = \lim_{x \to 0} \frac{-\sin x}{2x \cos x} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ form} \]

\[ = \lim_{x \to 0} \frac{-\cos x}{2 \cos x - 2x \sin x} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ form} \]

\[ = -\frac{1}{2} \]

\[ \Rightarrow \log y = -\frac{1}{2} \Rightarrow y = e^{-\frac{1}{2}} \]

**Example 3.10:** Find \( \lim_{x \to \pi/2^+} (\sec x)^{(\cot x)} \)

**Solution:** The given limit is of the form \( \infty^0 \)

So let \( y = (\sec x)^{(\cot x)} \)

\[ \Rightarrow \log y = \cot x \log (\sec x) \]
\[ \lim_{x \to \pi/2+0} \log y = \lim_{x \to \pi/2+0} \cot x \log (\sec x) \]

\[ = \lim_{x \to \pi/2+0} \cot x \log (\sec x) \]

\[ = \lim_{x \to \pi/2+0} \frac{\log (\sec x)}{\tan x} \]

\[ = \lim_{x \to \pi/2+0} \frac{\sec x \tan x}{\sec x} \]

\[ = \lim_{x \to \pi/2+0} \frac{\tan x}{\sec^2 x} \]

\[ = \lim_{x \to \pi/2+0} \frac{\sin x \times \cos^2 x}{\cos x} \]

\[ = 0 \]

\[ \Rightarrow \log y = 0 \]

\[ \Rightarrow y = e^0 = 1 \]

i.e. \( \lim_{x \to \pi/2+0} (\sec x)^{\cot x} = 1 \)

**Example 3.11:** Evaluate \( \lim_{x \to 0+0} x^x \)

**Solution:** It is of the form \(0^0\). So

\[ y = x^x \]

\[ \Rightarrow \log y = x \log x \]

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\[ \lim_{x \to 0^+} x^x = \lim_{x \to 0^+} x \log x \quad (0 \times \infty \text{ form}) \]

\[ = \lim_{x \to 0^+} \frac{\log x}{x^{-1}} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

\[ = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} \]

\[ = 0 \]

\[ \Rightarrow \log y = 0 \Rightarrow y = e^0 = 1 \]

**Exercise 3.3**

Evaluate the following limits:

1. \[ \lim_{x \to 0} (1 + x)^{1/x} \]
2. \[ \lim_{x \to 0^+} (\tan x)^{\sin 2x} \]
3. \[ \lim_{x \to 1} x^{(x-1)} \]
4. \[ \lim_{x \to 0} \left( \frac{5x + 1}{2x + 1} \right)^{1/x} \]
5. \[ \lim_{x \to a} \left( 3 - \frac{2x^2}{a^2} \right)^{\tan \left( \frac{\pi x}{2a} \right)} \]
6. \[ \lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x \], a being any non zero real number.

7. \[ \lim_{x \to \infty} \left(1 + \frac{1}{ax}\right)^x \], a being any non zero real number.

8. \[ \lim_{x \to \pi/2^+} (\tan x)^{\cos x} \]
Question 1: Examine the following function for maximum and minimum values:

\[ f(x) = x^5 - 5x^4 + 5x^3 - 1 \]

**Solution:**

Let \( f(x) = x^5 - 5x^4 + 5x^3 - 1 \)

We know that for extreme values \( f'(x) = 0 \)

i.e. \( 5x^4 - 20x^3 + 15x^2 = 0 \)

\[ \Rightarrow 5x^2(x - 1)(x - 3) = 0 \]

i.e. the critical points are \( x = 0, x = 1, x = 3 \)

Now \( f''(x) = 20x^3 - 60x^2 + 30x \)

\[ \Rightarrow f''(0) = 0, f''(1) = -10 \text{ and } f''(3) = 90 \]

i.e. \( f''(0) = 0, f''(1) < 0 \) and \( f''(3) > 0 \)

Thus, by theorem 1.3.5, \( f \) has maximum at \( x = 1 \) and minimum at \( x = 3 \) but the point \( x = 0 \) is to be examined further.

Now \( f'''(x) = 60x^2 - 120x + 30 \Rightarrow f'''(0) = 30 > 0 \)

i.e. \( f^{(n)}(0) \neq 0 \) for an odd value of \( n \)

Therefore, by theorem 1.3.8 (i), \( f \) has neither maximum nor minimum at \( x = 0 \).

Question 2: Find the maximum and minimum values of the polynomial

\[ 8x^5 - 15x^4 + 10x^2 \]

**Solution:**

Let \( f(x) = 8x^5 - 15x^4 + 10x^2 \)

We know that for extreme values \( f'(x) = 0 \)
i.e. $40x^4 - 60x^3 + 20x = 0$

$\Rightarrow 20x(2x^3 - 3x^2 + 1) = 0$

$\Rightarrow 20x(2x^3 - 2x^2 - x^2 + 1) = 0$

$\Rightarrow 20x(2x^2(x - 1) - (x^2 - 1)) = 0$

$\Rightarrow 20x(x - 1)(2x^2 - x - 1) = 0$

$\Rightarrow 20x(x - 1)^2(2x + 1) = 0$

$\Rightarrow$ The critical points are $x = 0, x = 1, x = -1/2$

Now $f''(x) = 160x^3 - 180x^2 + 20$

$\Rightarrow f''(0) = 20, f''(1) = 0$ and $f''(-1/2) = -45$

i.e. $f''(1) = 0$, $f''(-1/2) < 0$ and $f''(0) > 0$

Thus, by theorem 1.3.5, $f$ has maximum at $x = -1/2$ and minimum at $x = 0$ but the point $x = 1$ is yet to be examined.

Now $f'''(x) = 480x^2 - 360x \Rightarrow f'''(0) = 0$

$f'''(x) = 960x - 360 \Rightarrow f'''(0) < 0$

i.e. $f^{(n)}(0) \neq 0$ for an even value of $n$ and $f^{(n)}(0) < 0$

Therefore, by theorem 1.3.6(ii), $f$ has maximum at $x = 0$.

**Question 3:** Find the extreme points of the function in $[0, \pi]$

$$f(x) = \frac{1}{2} \sin x + \frac{1}{3} \sin 2x + \frac{1}{3} \sin 3x$$

**Solution:** For critical points $f'(x) = 0$

i.e. $\cos x + \cos 2x + \cos 3x = 0$

$\Rightarrow \cos 2x + \{\cos x + \cos 3x\} = 0$

$\Rightarrow \cos 2x + \{2 \cos 2x \cos x\} = 0$

$\Rightarrow \cos 2x \{1 + 2 \cos x\} = 0$
either \( \cos 2x = 0 \) or \( 1 + 2 \cos x = 0 \) or both the terms are simultaneously zero and \( x \in [0, \pi] \)

\[ 2x = \pi/2, 3\pi/2 \text{ or } \cos x = -1/2 \]

\[ x = \pi/4, 3\pi/4, 2\pi/3 \]

Now \( f''(x) = -\sin x - 2 \sin 2x - 3 \sin 3x \)

\[ f''(\pi/4) = -\frac{1}{\sqrt{2}} - 2 - \frac{3}{\sqrt{2}} < 0 \]

\[ f''(3\pi/4) = -\frac{1}{\sqrt{2}} + 2 - \frac{3}{\sqrt{2}} < 0 \]

\[ f''(2\pi/3) = -\frac{\sqrt{3}}{2} + 2 - \sqrt{3} = \frac{\sqrt{3}}{2} > 0 \]

Thus the function has maximum at two points i.e. at \( x = \pi/4 \) and \( 3\pi/4 \) while minimum at \( x = 2\pi/3 \).

**Question 4:** Prove that for \( 0 < x < 1 \)

\[ x < \sin^{-1} x < \frac{x}{\sqrt{1-x^2}} \]

**Solution:** Form two functions \( f(x) \) and \( g(x) \) as follows:

\[ f(x) = \sin^{-1} x - x \text{ and } g(x) = \sin^{-1} x - \frac{x}{\sqrt{1-x^2}} \]

Now \( f'(x) = \frac{1}{\sqrt{1-x^2}} - 1 \)

By hypothesis, \( x > 0 \implies x^2 > 0 \)

\[ -x^2 < 0 \implies 1 - x^2 < 1 \]
\[ \Rightarrow \sqrt{1 - x^2} < \sqrt{1} \]
\[ \Rightarrow \frac{1}{\sqrt{1 - x^2}} > 1 \]
\[ \Rightarrow 1 - \frac{1}{\sqrt{1 - x^2}} > 0 \]
\[ \Rightarrow f(x) > f(0) \]
\[ \Rightarrow \sin^{-1} x - x > \sin^{-1} 0 - 0 \]
\[ \Rightarrow \sin^{-1} x - x > 0 \]
\[ \Rightarrow \sin^{-1} x > x \]

Similarly, using the function \( g(x) \), the other half of the inequality can be proved.

**Question 5:** State Rolle’s theorem. Verify Rolle’s theorem for the function:

\[ f(x) = (x - a)^m (x - b)^n \text{ in } [a, b]; \text{ m, n being positive integers} \]

**Solution:** For the statement refer to Lesson 2. Let us verify the theorem.

(i) The function \( f \), being a polynomial in \( x \), is continuous in \([a, b]\).

(ii) \( f'(x) = (x - a)^{m-1}(x - b)^{n-1} \{ (m+n)x - (mb + na) \} \) which is defined in \( ]a, b[ \)

(iii) \( f(a) = 0 = f(b) \)

Thus all the conditions of Rolle’s theorem are satisfied.

Therefore \( \exists \) at least one \( c \in ]a, b[ \) such that \( f'(c) = 0 \)

\[ \Rightarrow f''(c) = (c - a)^{m-1} (c - b)^{n-1} \{ (m+n)c - (mb + na) \} = 0 \]
\[ \Rightarrow c = a, c = b, c = (mb + na) / (m+n) \]

As \( c \in ]a, b[ \), therefore, \( c \neq a \) and \( c \neq b \) but \( \{ (mb + na) / (m+n) \} \in ]a, b[ \)

Thus \( c = \frac{mb + na}{m+n} \) is the required value for the conclusion of the Rolle’s theorem.

Hence Rolle’s Theorem is verified.
**Question 6:** Show that for all $x > 0$

$$0 < [\log(1+x)]^{-1} - x^{-1} < 1$$

**Solution:** Let $f(x) = \log(1+x)$ and the interval is $[0,x]$

Then $f'(x) = 1/(x+1)$

Thus the $f(x)$ is continuous in $[0,x]$ and derivable in $[0,x]$.

∴ by Lagrange’s Mean value theorem $\exists$ at least one $c \in [0,x]$ such that

$$f(x) - f(0) = f'(c) = \frac{1}{x-0} (\log (1+x) - \log 1)$$

i.e. $$\frac{1}{1+c} = \frac{\log (1+x) - \log 1}{x}$$

i.e. $$\frac{1}{1+c} = \frac{\log (1+x)}{x}$$

…………………(I)

Now $$0 < c < x \Rightarrow 1 < 1 + c < 1 + x$$

$$\Rightarrow \frac{1}{1+x} < \frac{1}{1+c} < \frac{1}{1+x}$$

$$\Rightarrow [\log (1+x)]^{-1} < x^{-1} < 1$$

Also by (II)

$$\Rightarrow [\log (1+x)]^{-1} > x^{-1}$$

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\[ \Rightarrow [\log (1+x)]^{-1} - x^{-1} > 0 \quad \text{……………………(IV)} \]

Combining (III) and (IV), we have the required inequality.

**Question 7:** Evaluate:

\[ \frac{e^x - e^{-x} - 2 \log (1+x)}{x \sin x} \]

**Solution:** Let us simplify the expression at first.

Multiply and divide by \( x \), so that

\[
\frac{e^x - e^{-x} - 2 \log (1+x)}{x \sin x} = \frac{(e^x - e^{-x} - 2 \log (1+x))x}{x^2 \sin x} = \frac{e^x - e^{-x} - 2 \log (1+x)}{x^2} \left( \lim_{x \to 0} \frac{\sin x}{x} = 1 \right)
\]

Now, this expression is of 0/0 form. So, by L’ Hopital’s Rule, we have

\[
= \lim_{x \to 0} \frac{e^x - (-1)e^{-x} - 2/(1+x)}{2x} = \lim_{x \to 0} \frac{e^x + e^{-x} - 2/(1+x)}{2x} \quad \text{----- form}
\]

\[
= \lim_{x \to 0} \frac{(e^x - e^{-x}) + 2/(1+x)^2}{2} = 1
\]
Question 8: Find the values of p and q for which
\[
\lim_{x \to 0} \frac{x (1 + p \cos x) - q \sin x}{x^3}
\]
exists and equals 1.

Solution: Since expression on the left hand side is of 0 / 0 form, so applying L’ Hopital’s Rule, we have
\[
\lim_{x \to 0} \frac{x (1 - p \sin x) - q \sin x}{x^3} = \lim_{x \to 0} \frac{1 + p \cos x + x (-p \sin x) - q \cos x}{3x^2}
\]
For the further application of the L’ Hopital’s rule we should have limit of numerator as well as of denominator to be zero, separately. Thus, we have the first equation as
\[
1 + p \cos 0 + 0 (-p \sin 0) - q \cos 0 = 0
\]
i.e. 1 + p - q = 0…………………………………………………..(I)
Assuming (I) to be true, we can apply L’ Hopital’s rule again. Hence we have
\[
\lim_{x \to 0} \frac{-2p \sin x - px \cos x + q \sin x}{6x} = \lim_{x \to 0} \frac{-px \cos x + (q - 2p) \sin x}{6x} \tag{0 form}
\]
\[
\lim_{x \to 0} \frac{-p \cos x + px \sin x + (q - 2p) \cos x}{6} \tag{II}
\]
By the statement of the problem, the limit of (II) is 1. Thus we have another relation as
\[
\frac{-p \cos 0 + 0 \cdot (p \sin 0) + (q - 2p) \cos 0}{6} = 1
\]
i.e. \(-3p + q = 6\)  

Solving (I) and (III), we get \(p = -5/2\) and \(q = -3/2\).

**Question 9:** Evaluate:
\[
\lim_{x \to 0} \frac{\tan^2 x - x^2}{x^2 \tan x}
\]

**Solution:** Let us simplify the expression at first. Multiply and divide by \(x^2\), so that
\[
\lim_{x \to 0} \frac{(\tan^2 x - x^2) x^2}{x^4 \tan^2 x}
\]

\[
= \lim_{x \to 0} \frac{\tan^2 x - x^2}{x^4} \cdot \frac{x^2}{\tan^2 x}
\]

\[
= \lim_{x \to 0} \frac{\tan^2 x - x^2}{x^4} \left(\lim_{x \to 0} \frac{\tan x}{x} = 1\right)
\]

Now, this expression is of \(0 / 0\) form. So, by L’ Hopital’s Rule, we have
\[
= \lim_{x \to 0} \frac{\tan x \sec^2 x - x}{2x^3} \left(\begin{array}{c} 0 \\ 0 \end{array}\right)
\]

\[
= \lim_{x \to 0} \frac{2 \tan^2 x \sec^2 x + \sec^4 x - 1}{6x^2} \left(\begin{array}{c} 0 \\ 0 \end{array}\right)
\]
\[
\begin{align*}
\lim_{x \to 0} \frac{2 \tan x \sec^4 x + \tan^3 x \sec^2 x}{3x} &= \frac{0}{0} \\
\lim_{x \to 0} \frac{8 \tan^2 x \sec^4 x + 2 \sec^6 x + 3 \tan x \sec^4 x + 2 \tan^4 x \sec^2 x}{3} &= \lim_{x \to 0} \frac{3}{3} = \frac{2}{3}
\end{align*}
\]

**Question 10:** Using Maclaurin’s series expansion of \(\log(1 + x)\) and \(e^x\), evaluate
\[
\lim_{x \to 0} \frac{(1 + x)^{1/x} - e + ex/2}{x^2}
\]

**Solution:** The Maclaurin’s expansion for \(\log(1 + x)\) and \(e^x\) are given by
\[
\log(1 + x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \ldots \quad \text{for } -1 < x \leq 1 \quad \text{(I)}
\]
and
\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \quad \text{for all } x \in \mathbb{R} \quad \text{(II)}
\]

Let \(y = (1 + x)^{1/x}\)

\[
\Rightarrow \log y = (1/x) \log(1 + x) = \frac{1}{x} \left( x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \ldots \right) \quad \text{(by (I))}
\]
\[
= 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots
\]
\[
\Rightarrow y = e^{z+1} = e \cdot e^z
\]

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\[= e \left[ 1 - \frac{z}{2!} + \frac{z^2}{3!} - \frac{z^3}{4!} + \ldots \right] \text{ (by (II))} \]

\[= e \left[ 1 - \frac{1}{2!} \left\{ -\frac{x}{2!} + \frac{x^2}{3!} - \frac{x^3}{4!} + \ldots \right\} + \right. \]

\[\left. - \frac{1}{3!} \left\{ -\frac{x}{2!} + \frac{x^2}{3!} - \frac{x^3}{4!} + \ldots \right\}^2 \right] \]

\[= e \left[ 1 - \frac{x}{2} + \frac{11}{24} x^2 - \ldots \right] \]

\[
\therefore \quad \lim_{x \to 0} \frac{(1 + x)^{1/x} - e + ex/2}{x^2} = \left\{ \begin{array}{c}
1 - \frac{x}{2} + \frac{11}{24} x^2 - \ldots \end{array} \right\} - e + \frac{ex}{2} \]

\[= \lim_{x \to 0} \left( e + \text{terms containing } x \right) \]

\[= \frac{11}{24} e \]
MISCELLANEOUS EXERCISE

**Question 1:** Divide 100 into two parts such that the sum of their squares is minimum.

**Solution:** Let x and y be two parts of 100. Then we can write

\[ x + y = 100 \]
\[ \Rightarrow y = 100 - x \]

As we have to minimize the sum of the squares of these two parts, so we have the function \( f(x) \) as

\[ f(x) = x^2 + (100 - x)^2 \]

For extreme values, \( f'(x) = 0 \)
\[ \Rightarrow 2x + 2(100 - x) (-1) = 0 \]
\[ \Rightarrow x - 100 + x = 0 \]
\[ \Rightarrow x = 50 \]

Also \( f''(x) = 2 \) for all values of \( x \) and hence for \( x = 50 \)

Thus, \( f(x) \) has minimum at \( x = 50 \).

For \( x = 50 \), \( y = 100 - x = 100 - 50 = 50 \)

i.e. the sum of the squares of the two parts of 100 is minimum if both parts are of the same magnitude i.e. equal to 50 each.

**Question 2:** Show that \( y = x + x^{-1} \) has two points of extrema, one of maximum and the other of minimum and the value of the latter is larger than the former.

**Solution:** Here

\[ f(x) = x + \frac{1}{x} \]

For extreme values, \( f'(x) = 0 \)
\[ f'(x) = 1 - \frac{1}{x^2} = 0 \]

\[ \Rightarrow x^2 - 1 = 0 \text{ i.e. the critical points are } x = 1 \text{ and } x = -1 \]

Now \( f''(x) = \frac{2}{x^3} \) which is positive for \( x = 1 \) and negative for \( x = -1 \)

\[ \Rightarrow f(x) \text{ has maximum at } x = -1 \text{ and the maximum value is } f(-1) = -1 + (-1)^{-1} = -2 \]

and the function \( f(x) \) has minimum at \( x = 1 \) and the minimum value is \( f(1) = 2 \).

Obviously (but strangely), the maximum value of the function is less than the minimum value of the function, which was to be verified in the problem.

**Question 3:** Show that \( f(x) = \frac{(x + 1)^2}{(x + 3)^3} \) has a maximum value \( \frac{2}{27} \) and a minimum value \( 0 \).

**Question 4:** Find the inflection points, if any.

(i) \[ f(x) = 3x^4 + 4x^3 - 12x^2 + 2 \]

(ii) \[ f(x) = \sin x \text{ on } [0, 2\pi] \]

(iii) \[ f(x) = x^4 \]

**Solution:**

(i) \[ f'(x) = 12x^3 + 12x^2 - 24x = 0 \]

\[ \Rightarrow x^3 + x^2 - 2x = 0 \]

\[ \Rightarrow x(x^2 + x - 2) = 0 \]

\[ \Rightarrow x(x - 1)(x + 2) = 0 \]

\[ \Rightarrow x = 0, 1, -2 \text{ are the critical points.} \]

For inflection points \( f''(x) = 0 \)

\[ \text{i.e. } 36x^2 + 24x - 24 = 0 \]

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\[ 3 x^2 + 2 x - 2 = 0 \]

\[ x = (-1\pm\sqrt{7}) / 3 \approx -1.22, 0.55 \]

Now \( f'''(x) = 72 x + 24 \neq 0 \) for \( x = 1.22 \) and 0.55

Thus the points of inflection are \( x = -1.22 \) and 0.55

(ii) \( f'(x) = \cos x \Rightarrow f''(x) = -\sin x \Rightarrow f'''(x) = -\cos x \)

Now \( f''(x) = 0 \Rightarrow x = 0, \pi, 2\pi \).

As 0 and 2\( \pi \) are the end points so the only inflection point is \( x = \pi \).

(iii) \( f'(x) = 4x^3 \Rightarrow f''(x) = 12x^2 \Rightarrow f'''(x) = 24x \)

Now \( f''(x) = 0 \Rightarrow x = 0 \).

But \( f'''(x) = 0 \) for \( x = 0 \), therefore there is no inflection point.

**Question 5:** Prove that for a general cubic polynomial

\[ f(x) = ax^3 + bx^2 + cx + d \], \( (a \neq 0) \)

has exactly one inflection point.

**Solution:** \( f'(x) = 3ax^2 + 2bx + c = 0 \)

\[ \Rightarrow f''(x) = 6ax + 2b = 0 \]

\[ -b \]

\[ x = \frac{3a}{3a} \]

Now \( f'''(x) = 6a \neq 0 \) for all \( x \) and hence for \( x = \frac{-b}{3a} \)

Thus the point of inflection is only one, provided \( a \neq 0 \)

**Question 6:** Find the relative extrema:

(i) \( f(x) = 2x^3 - 9x^2 + 12x \)

(ii) \( f(x) = -x^2 - 4x + 1 \)
(iii) \( f(x) = \sin^2 x, \ 0 < x < 2\pi \)
(iv) \( f(x) = \frac{1}{2} x - \sin x, \ 0 < x < 2\pi \)
(v) \( f(x) = x^3 + 5x - 2 \)
(vi) \( f(x) = x^4 - 2x^2 + 7 \)
(vii) \( f(x) = x (x - 1)^2 \)
(viii) \( f(x) = x^4 + 2x^3 \)

**Question 7:** Evaluate:

(i) \( \lim_{x \to 0^+} x \log x \)

(ii) \( \lim_{x \to \pi/4} (1 - \tan x) \sec 2x \)

(iii) \( \lim_{x \to \pi^+} \frac{\sin x}{(x - \pi)} \)

(iv) \( \lim_{x \to \infty^+} \frac{x^{100}}{e^x} \)

(v) \( \lim_{x \to 0^+} \frac{\log (\sin x)}{\log (\cos x)} \)

(vi) \( \lim_{x \to 0} \frac{\sin^{-1} 2x}{x} \)

(vii) \( \lim_{x \to \pi^-} (x - \pi) \tan(x/2) \)

(viii) \( \lim_{x \to +\infty} \left\{ x - \log(x^2 + 1) \right\} \)
Question 8: Why L’ Hopital’s rule does not apply to the problem?

\[
\lim_{x \to 0} \frac{x^2 \sin (1/x)}{\sin x}
\]

Solution: We can rewrite the given expression as

\[
\lim_{x \to 0} \frac{x \sin (1/x)}{(\sin x)/x}
\]

\[= \lim_{x \to 0} \frac{x \sin (1/x)}{1} \quad \text{because } \lim (\sin x)/x = 1
\]

\[= \lim_{x \to 0} x \sin (1/x)
\]

since \( \sin (1/x) \) does not exist for \( x = 0 \), we can’t find out the limit of the numerator and denominator separately to get a 0/0 form. Therefore L’Hopital’s rule is not applicable.

Question 9: Find all values of A and B such that

\[
\lim_{x \to 0} \frac{A + \cos Bx}{x^2} = -4
\]

Question 10: Establish the following inequalities for \( x > 0 \):

(i) \( x < \log \left( \frac{1}{1-x} \right) < \frac{x}{1-x} \)

(ii) \( x < e^x - 1 < \frac{x}{1-x} \)

(iii) \( x^3 - 6x^2 + 15x + 3 > 0 \)